CHAPTER XIV: RISK MODELING

Risk is often cited as a factor which influences decisions. This chapter reviews methods for incorporating risk and risk reactions into mathematical programming models. [1]

Mathematical programming risk models depict the risk inherent in model parameters. Risk considerations are usually incorporated assuming that the parameter probability distribution (i.e., the risk) is known with certainty. [2] Usually, the task becomes one of adequately representing these distributions as well as the decision makers response to parameter risk.

The question arises: Why model risk, why not just solve the model under all combinations of the risky parameters and use the resultant plans? Such an approach is tempting, yet suffers from problems of dimensionality and certainty. The dimensionality problem is manifest in the number of possible plans; (i.e., five possible values for each of three parameters would lead to $3^5 = 243$ possible parameter specifications). Often, there are more possible states of nature than can practically be enumerated. Furthermore, these enumerated plans suffer from a certainty problem. Every LP parameter is assumed known with perfect knowledge. Consequently, solutions reflect "certain" knowledge of the parameter values imposed. Thus, when one solves many models one gets many plans and the question remains which plan should be used.

Usually, it is desirable to generate a robust solution which yields satisfactory results across the distribution of parameter values. The risk modeling techniques discussed below are designed to yield such a plan. The "optimal" plan for a risk model generally does not place the decision maker in the best possible position for all (or maybe even any) possible events, but rather establishes a robust position across the set of possible events.

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[1] The risk modeling problem is a form of the multiple objective programming problem so that there are parallels between the material here and that in the multi-objective chapter.

[2] It should be noted that risk and uncertainty are used interchangeably. Any time we discuss risk or uncertainty we assume that the probability distribution is known.

14.1 Decision Making and Recourse

Many different programming formulations have been posed for risk problems. An important assumption involves the potential decision maker reaction to information. The most fundamental distinction is between cases where:

1) all decisions must be made now with the uncertain outcomes resolved later, after all random draws from the distribution have been taken, and

2) some decisions are made now, then later some uncertainties are resolved followed by other decisions yet later.

These two settings are illustrated as follows. In the first case, all decisions are made then events occur and outcomes are realized. This is akin to a situation where one invests now and then discovers the returns to the investment at year end without any intermediate buying or selling decisions. In the second case, one makes some decisions now, gets some information and makes subsequent decisions. Thus, one might invest at the beginning of the year, but could sell and buy during the year depending on changes in stock prices.

The main distinction is that under the first situation decisions are made before any uncertainty is resolved and no decisions are made after any of the uncertainty is resolved. In the second situation, decisions are made sequentially with some decisions made conditional upon outcomes that were subject to a probability distribution at the beginning of the time period.

These two frameworks lead to two very different types of risk programming models. The first type of model is most common and is generally called a stochastic programming model. The second type of model was originally developed by Dantzig in the early 50's and falls into the class of stochastic programming with recourse models. These approaches are discussed separately, although many stochastic programming techniques can be used when dealing with stochastic programming with recourse problems.

14.2 An Aside: Discounting Coefficients

Before discussing formal modeling approaches, first let us consider a common, simple approach to risk used in virtually all "risk free" linear programming studies. Suppose a parameter is distributed according to some probability distribution, then a naive risk specification would simply use the mean. However, one could also use conservative price estimates (i.e., a price that one feels will be exceeded 80% of the time).

This reveals a common approach to risk. Namely, data for LP models are virtually never certain. Conservative estimates are frequently used, in turn producing conservative plans (see McCarl et al., for an example of treatment of time available). Objective function revenue coefficients may be deflated while cost coefficients are inflated. Technical coefficients and right hand sides may be treated similarly. The main difficulty with a conservative estimate based approach is the resultant probability of the solution. Conservative estimates for all parameters can imply an extremely unlikely event and an overly conservative choice of the decision variables.

14.3 Stochastic Programming without Recourse

Stochastic programming techniques generally treat risk in the objective function coefficients, technical coefficients or right hand sides separately or collectively.

14.3.1 Objective Function Coefficient Risk

Several objective function coefficient risk models have been posed. This section reviews these. First, however, some statistical background on distributions of linear sums is necessary.

Given a linear objective function

$$Z = c_1 X_1 + c_2 X_2$$

where X_1, X_2 are decision variables and c_1, c_2 are uncertain parameters distributed with means $\overline{c}_1$ and $\overline{c}_2$ as well as variances s_{11}, s_{22}, and covariance s_{12}; then Z is distributed with mean

$$\overline{Z} = \overline{c_1} X_1 + \overline{c_2} X_2$$

and variance

$$\sigma_Z^2 = s_{11} X_1^2 + s_{22} X_2^2 + 2 s_{21} X_1 X_2.$$

In matrix terms the mean and variance of Z are

$$(\overline{C}X , X^{/}SX)$$

where in the two by two case

$$\overline{C} = \left[\overline{c_1} \ \overline{c_2} \right] \qquad S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} .$$

Defining terms

s_{ii} is the variance of the objective function coefficient of X_i, which is calculated using the formula $s_{ik} = \sum (c_{ik} - \overline{c_i})^2/N$ where c_{ik} is the k^{th} observation on the objective value of X_i and N is the number of observations.[3]

s_{ij} for $i \neq j$ is the covariance of the objective function coefficients between X_i and X_j, calculated by the formula $s_{ij} = \sum (c_{ik} - \overline{c_i})(c_{jk} - \overline{c_j})/N$. Note $s_{ij} = s_{ji}$.

$\overline{c_i}$ is the mean value of the objective function coefficient associated with X_i, calculated by $\overline{c_i} = \sum c_{ik}/N$. (Assuming an equally likely probability of occurrence.)

14.3.1.1 Mean-Variance Analysis

The above expressions define the mean and variance of a LP objective function with risky c parameters. Markowitz exploited this in the original mean-variance portfolio choice formulation.

The portfolio choice problem involves development of an "optimal" investment strategy. The variables indicate the amount of funds invested in each risky investment subject to a total funds constraint. Markowitz motivated the formulation by observing that investors only place a portion, not all, of their funds in the highest-yielding investment. This, he argued, indicated that a LP formulation is inappropriate since such an LP would reflect investment of all funds in the highest yielding alternative (since there is a single constraint). This divergence between observed and modeled behavior led Markowitz to include a variance

[3] One could also use the divisor N-1 when working with a sample.

term resulting in the so-called expected value variance (E-V) model.

Freund (1956) developed a related model, apparently independently, which has become the most commonly used E-V model. The portfolio context of his formulation is

$$
\begin{array}{llll}
\text{Max} & \sum_j \bar{c}_j X_j - \Phi \sum_j \sum_k s_{jk} X_j X_k & & \\
\text{s.t.} & \sum_j X_j & = & 1 \\
 & X_j & \geq & 0 \quad \text{for all } j
\end{array}
$$

Here the objective function maximizes expected income ($\bar{c}X$) less a "risk aversion coefficient" (Φ) times the variance of total income (X'SX). The model assumes that decision makers will trade expected income for reduced variance.

In this context Markowitz discussed the E-V efficient frontier which is the locus of points exhibiting minimum variance for a given expected income, and/or maximum expected income for a given variance of income (Figure 14.1 gives the frontier for the example below). Such points are efficient for a decision maker with positive preference for income, negative preference for variance and indifference to other factors.

The E-V problem can handle problem contexts broader than the portfolio example. A general formulation in the resource allocation context is

$$
\begin{array}{llll}
\text{Max} & \bar{C}X - \Phi X'SX & & \\
\text{s.t.} & AX & \leq & b \\
 & X & \geq & 0
\end{array}
$$

where $\bar{C}$ is average returns from producing X and S gives the associated variance-covariance matrix.

14.3.1.1.1 Example

Assume an investor wishes to develop a stock portfolio given the stock annual returns information shown in Table 14.1, 500 dollars to invest and prices of stock one $22.00, stock two $30.00, stock three $28.00 and stock four $26.00.

The first stage in model application is to compute average returns and the variance-covariance matrix of total net returns. The mean returns and variance - covariance matrix are shown in Table 14.2. In turn the objective function is

$$\text{Max} \ [\ 4.70 \ \ 7.60 \ \ 8.30 \ \ 5.80 \ ] \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} - \Phi \ [X_1 \ \ X_2 \ \ X_3 \ \ X_4] \begin{bmatrix} +3.21 & -\ 3.52 & +\ 6.99 & +0.04 \\ -3.52 & +\ 5.84 & -13.68 & +0.12 \\ +6.99 & -13.68 & +61.81 & -1.64 \\ +0.04 & +\ 0.12 & -\ 1.64 & +0.36 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

or, in scaler notation

$$\begin{array}{ll} \text{Max} & 4.70\ X_1 \ + \ 7.60\ X_2 \ + \ 8.30\ X_3 \ + \ 5.80\ X_4 \\ - \ \Phi & \left(\begin{array}{llll} +\ 3.21\ \ X_1^2 & -\ 3.52\ X_1X_2 & +\ 6.99\ X_1X_3 & +\ 0.04\ X_1X_4 \\ -\ 3.52\ X_2X_1 & +\ 5.84\ \ X_2^2 & -\ 13.68\ X_2X_3 & +\ 0.12\ X_2X_4 \\ +\ 6.99\ X_3X_1 & -\ 13.68\ X_3X_2 & +\ 61.81\ \ X_3^2 & -\ 1.64\ X_3X_4 \\ +\ 0.04\ X_4X_1 & +\ 0.12\ X_4X_2 & -\ 1.64\ X_4X_3 & +\ 0.36\ \ X_2^4 \end{array} \right) \end{array}$$

This objective function is maximized subject to a constraint on investable funds:

$$22X_1 \ + \ 30X_2 \ + \ 28X_3 \ + \ 26X_4 \ \leq \ 500$$

and non-negativity conditions on the variables.

Empirically, this problem is solved for various Φ values as implemented in the GAMS instructions in Table 14.3 or in the EVPORTFO file. The solutions, at selected values of Φ, are shown in Table 14.4, while Figure 14.1 gives the efficient frontier.

The model yields the profit maximizing solution ($X_1=X_2=X_4=0, X_3=17.86$) for low risk aversion parameters ($\Phi < 0.0005$). As the risk aversion parameter increases, then X_2 comes into the solution. The simultaneous use of X_2 and X_3 coupled with their negative covariance reduces the variance of total returns. This pattern continues as Φ increases. For example, when Φ equals 0.012 expected returns have fallen by \$17 or 11%, while the standard deviation of total returns has fallen by \$117 or 80%. For yet higher values of the risk aversion parameter, investment in X_1 increases, then later X_4 is added.

Three other aspects of these results are worth noting. First, the shadow price on investable capital continually decreases as the risk aversion parameter (Φ) increases. This reflects an increasing risk discount as risk aversion increases. Second, solutions are reported only for selected values of Φ. However, any change in Φ leads to a different solution and an infinite number of alternative Φ's are possible; e.g., all solutions between Φ values of 0.0005 and 0.0075 are convex combinations of those two solutions. Third, when Φ becomes sufficiently large, the model does not use all its resources. In this particular case, when Φ exceeds 2.5, not all funds are invested.

14.3.1.1.2 Markowitz's E-V Formulation

Markowitz's original formulation of the E-V problem minimized variance subject to a given level of expected income as in the multi-objective programming lexicographic formulation.

Algebraically, this model is

$$
\begin{array}{lrcl}
\text{Min} & X'SX & & \\
\text{s.t.} & \bar{C}X & = & \lambda \\
 & AX & \leq & b \\
 & X & \geq & 0
\end{array}
$$

where λ is parameterized over the relevant part of the range of possible expected incomes i.e. from the lowest acceptable to the LP maximum.

14.3.1.1.3 Formulation Choice

Markowitz's (1959) and Freund's (1956) formulations yield identical efficient frontiers; however, we favor Freund's (1956) formulation (a weighted multi-objective tradeoff model) due to a perceived incompatibility of the Markowitz formulation with model use as argued in the multi-objective chapter. Briefly, models are usually formulated for comparative statics analysis of a related series of problems. This type of analysis involves changes in the S, $\bar{C}$, A and b parameters. In such an analysis, we feel it is not desirable to give alternative efficient frontiers; rather, we feel it is desirable to give specific plans (i.e., X variable values) for the S, $\bar{C}$, A and b settings. Using the above E-V models one would first need to select either a numerical value for Φ or one for λ. A value of Φ so adopted is largely a function of the decision makers' preference between income and risk (see Freund (1956) or Bussey for theoretical development of this point). The value of λ adopted will be a function of both the risk-income tradeoff and the values of $\bar{C}$, S, A, and b. Thus, the attainability of a given choice λ would change with alterations in these parameters. On the other hand, Φ expresses a "pure" measure of the risk-tradeoff and is more likely to be relevant for different parameter values. Thus, we prefer the Freund (1956) formulation.

14.3.1.1.4 Characteristics of E-V Model Optimal Solutions

Properties of optimal E-V solutions may be examined via the Kuhn-Tucker conditions. Given the problem

$$\begin{array}{lll} \text{Max} & \bar{C}X - \Phi X'SX & \\ \text{s.t.} & AX & \leq b \\ & X & \geq 0 \end{array}$$

Its Lagrangian function is

$$\mathcal{L}(X,\mu) = \bar{C}X - \Phi X'SX - \mu(AX - b)$$

and the Kuhn-Tucker conditions are

$$\begin{array}{rcccc} \partial\mathcal{L}/\partial X & = & \bar{C} - 2\phi X'S - \mu A & \leq & 0 \\ (\partial\mathcal{L}/\partial X)X & = & (\bar{C} - 2\phi X'S - \mu A)X & = & 0 \\ X & & & \geq & 0 \\ \partial\mathcal{L}/\partial\mu & = & -(AX - b) & \geq & 0 \\ \mu(\partial\mathcal{L}/\partial\mu) & = & \mu(AX - b) & = & 0 \\ \mu & & & \geq & 0 \end{array}$$

where μ is the vector of dual variables (Lagrangian multipliers) associated with the primal constraint $AX \leq b$.

A cursory examination of these conditions indicates two things. First, the solution permits more variables to be nonzero than would a LP basic solution. This occurs since variables can be nonzero to satisfy the n potential conditions $\partial\mathcal{L}/\partial X = 0$ and the m conditions where AX = b or $\mu = 0$. Thus, the solution can have more nonzero variables than constraints. Second, the $\partial\mathcal{L}/\partial X$ equation relates resource cost (μ) with marginal revenue ($\bar{C}$) and a marginal cost of bearing risk ($-2\Phi X'S$). Consequently, the optimal shadow prices are risk adjusted as are the optimal decision variable values.

14.3.1.1.5 E-V Model Use - Theoretical Concerns

Use of the E-V model has been theoretically controversial. Expected utility theory (von Neumann and Morgenstern) provides the principal theoretical basis for choice under uncertainty. Debate has raged, virtually since the introduction of E-V analysis, on the conditions under which an E-V model makes choices equivalent to expected utility maximization. Today the general agreement is that maximizing the E-V problem is equivalent to maximizing expected utility when one of two conditions hold: 1) the underlying income distribution is normal - which requires a normal distribution of the c_j and the utility function is exponential (Freund, 1956; Bussey) [4], and 2) the underlying distributions satisfy Meyer's location and scale restrictions. In addition, Tsiang (1972, 1974) has shown that E-V analysis provides an acceptable approximation of the expected utility choices when the risk taken is small relative to total initial wealth. The E-V frontier has also been argued to be appropriate under quadratic utility (Tobin). There have also been empirical studies (Levy and Markowitz; Kroll, et al.; and Reid and Tew) wherein the closeness of E-V to expected utility maximizing choices has been shown.

14.3.1.1.6 Specification of the Risk Aversion Parameter

E-V models need numerical risk aversion parameters (Φ). A number of approaches have been used for parameter specification. First, one may avoid specifying a value and derive the efficient frontier. This involves solving for many possible risk aversion parameters. Second, one may derive the efficient frontier

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[4] Normality probably validates a larger class of utility functions but only the exponential case has been worked out.

and present it to a decision maker who picks an acceptable point (ideally, where his utility function and the E-V frontier are tangent) which in turn identifies a specific risk aversion parameter (Candler and Boeljhe). Third, one may assume that the E-V rule was used by decision makers in generating historical choices, and can fit the risk aversion parameter as equal to the difference between marginal revenue and marginal cost of resources, divided by the appropriate marginal variance (Weins). Fourth, one may estimate a risk aversion parameter such that the difference between observed behavior and the model solution is minimized (as in Brink and McCarl (1979) or Hazell et al. (1983)). Fifth, one may subjectively elicit a risk aversion parameter (see Anderson, et al. for details) and in turn fit it into the objective function (i.e., given a Pratt risk aversion coefficient and assuming exponential utility implies the E-V Φ equals 1/2 the Pratt risk aversion coefficient [Freund, 1956 or Bussey]). Sixth, one may transform a risk aversion coefficient from another study or develop one based on probabilistic assumptions (McCarl and Bessler).

The E-V model has a long history. The earliest application appears to be Freund's (1956). Later, Heady and Candler; McFarquhar; and Stovall all discussed possible uses of this methodology. A sample of applications includes those of Brainard and Cooper; Lin, et al.; and Wiens. In addition, numerous references can be found in Boisvert and McCarl; Robinson and Brake; and Barry.

14.3.1.2 A Linear Approximation - MOTAD

The E-V model yields a quadratic programming problem. Such problems traditionally have been harder to solve than linear programs (although McCarl and Onal argue this is no longer true). Several LP approximations have evolved (Hazell, 1971; Thomas et al; Chen and Baker; and others as reviewed in McCarl and Tice). Only Hazell's MOTAD is discussed here due to its extensive use.

The acronym MOTAD refers to Minimization of Total Absolute Deviations. In the MOTAD model, absolute deviation is the risk measure. Thus, the MOTAD model depicts tradeoffs between expected income and the absolute deviation of income. Minimization of absolute values is discussed in the nonlinear approximations chapter. Briefly reviewing, absolute value may be minimized by constraining the terms whose absolute value is to be minimized (D_k) equal to the difference of two non-negative variables ($D_k = d_k^+ - d_k^-$) and in turn minimizing the sum of the new variables $\sum (d_k^+ + d_k^-)$. Hazell(1971) used this formulation in developing the MOTAD model. [5]

Formally, the total absolute deviation of income from mean income under the k^{th} state of nature (D_k) is

$$D_k = \left| \left(\sum_j c_{kj} X_j \right) - \left(\sum_j \bar{c}_j X_j \right) \right|$$

where c_{kj} is the per unit net return to X_j under the k^{th} state of nature and $\bar{c}_j$ is the mean.

Since both terms involve X_j and sum over the same index, this can be rewritten as

$$D_k = \left| \sum_j (c_{kj} - \bar{c}_j) X_j \right|$$

Total absolute deviation (TAD) is the sum of D_k across the states of nature. Now introducing deviation variables to depict positive and negative deviations we get

$$TAD = \sum_k D_k = \sum_k (d_k^+ + d_k^-)$$

$$\text{where } \sum_j (c_{kj} - \bar{c}_j) X_j - d_k^+ + d_k^- = 0 \text{ for all } k$$

Then adding the sum of the deviation variables to the objective function the MOTAD model maximizes expected net returns less a risk aversion coefficient (Ψ) times the measure of absolute deviation. The final MOTAD formulation is

$$\begin{array}{llllll}
\text{Max} & \sum_j \bar{c}_j X_j & - \psi \sum_k (d_k^+ + d_k^-) & & & \\
\text{s.t.} & \sum_j (c_{kj} - \bar{c}_j) X_j & - d_k^+ + d_k^- & = & 0 & \text{for all } k \\
 & \sum_j a_{ij} X_j & & \leq & b_i & \text{for all } i \\
 & X_j , & d_k^+ , d_k^- & \geq & 0 & \text{for all } j,k
\end{array}$$

5 The approach was suggested in Markowitz (1959, p. 187).

where d_k^+ is the positive deviation of the k^{th} income occurrence from mean income and d_k^- is the associated negative deviation. [6]

There have been a number of additional developments regarding the MOTAD formulation. Hazell formulated a model considering only negative deviations from the mean, ignoring positive deviations. This formulation is:

$$
\begin{array}{llllll}
\text{Max} & \sum_j \bar{c}_j X_j & - \theta \sum_k d_k^- & & & \\
\text{s.t.} & \sum_j (c_{kj} - \bar{c}_j) X_j & + \quad d_k^- & \geq & 0 & \text{for all } k \\
& \sum_j a_{ij} X_j & & \leq & b_i & \text{for all } i \\
& X_j, & \quad d_k^- & \geq & 0 & \text{for all } j,k
\end{array}
$$

However, Hazell notes that when the deviations are taken from the mean, the solution to this problem is equivalent to the total absolute value minimization where $\theta = 2\Psi$ due to the symmetry of the deviations. The negative deviations only model is the more commonly used MOTAD formulation (for example, see Brink and McCarl).

Also, Hazell (1971) reviews Fisher's development which shows that the standard error of a normally distributed population can be estimated given sample size N, by multiplying mean absolute deviation (MAD), total absolute deviation (TAD), or total negative deviation (TND) by appropriate constraints. Thus,

$$
\sigma \approx \left|\frac{\pi \; N}{2\;(N-1)}\right|^{0.5} MAD = \left|\frac{\pi \; N}{2\;(N-1)}\right|^{0.5} \frac{TAD}{N} = \left|\frac{\pi}{2\;N\;(N-1)}\right|^{0.5} TAD = \left|\frac{2\pi}{N\;(N-1)}\right|^{0.5} TND
$$

where π = 22/7 or 3.14176.

This transformation is commonly used in MOTAD formulations. A formulation incorporates such as

[6] Note this formulation approach can be used within an E-V framework if one squares d^+ and d^- in the objective function.

$$
\begin{array}{llllllll}
\text{Max} & \sum_j \bar{c}_j X_j & & & & - \gamma\ \sigma & & \\
\text{s.t.} & \sum_j (c_{kj} - \bar{c}_j)\ X_j & & + d_k^- & & & \geq 0 & \text{for all } k \\
 & \sum_j a_{ij}\ X_j & & & & & \leq b_i & \text{for all } i \\
 & & -TND & + \sum_k d_k^- & & & = 0 & \\
 & & \left(\frac{2\pi}{N\ (N-1)} \right)^{0.5} TND & & & - \sigma & = 0 & \\
 & X_j, & TND, & d_k^-, & & \sigma & \geq 0 & \text{for all } j,k
\end{array}
$$

14.3.1.2.1 Example

This example uses the same data as in the E-V Portfolio example. Deviations from the means (c_{kj} - $\bar{c}_j$) for the stocks are shown in Table 14.5. The MOTAD formulation is given in Table 14.6. The equivalent GAMS statement is called MOTADPOR.

Here Δ is the constant which approximates standard error from the empirical value of TND as discussed above. This problem is solved for over a range of values for γ. The associated solutions are reported in Table 14.7 and contain information on investment in the nonzero X_j's, unused funds, mean absolute deviation, and the approximation of the standard error. Also, the true variance and standard error are calculated from the solution values and the original data. Note the approximate nature of the Fisher standard error formula. For example, the approximated standard error at the first risk aversion range is 161.4, but the actual standard error is 140.4. The approximation initially overstates the true standard error, but later becomes quite close. The E-V and MOTAD frontiers correspond closely (see Figure 14.2). However, this is not adequate proof that the solutions will always be close (see Thomson and Hazell for a comparison between the methods).

14.3.1.2.2 Comments on MOTAD

Many of the E-V model comments are appropriate here and will not be repeated. However, a number of other comments are in order. First, a cursory examination of the MOTAD model might lead one to conclude covariance is ignored. This is not so. The deviation equations add across all the variables, allowing negative deviation in one variable to cancel positive deviation in another. Thus, in minimizing total absolute deviation the model has an incentive to "diversify", taking into account covariance.

Second, the equivalence of the total negative and total absolute deviation formulations depends critically upon deviation symmetry. Symmetry will occur whenever the deviations are taken from the mean. This, however, implies that the mean is the value expected for each observation. This may not always be the case. When the value expected is not the mean, then moving averages or other expectation models should be used instead of the mean (see Brink and McCarl, or Young). In such cases, the deviations are generally non-symmetric and consideration must be given to an appropriate measure of risk. For example, Brink and McCarl use a mean negative deviation formulation with a moving average expectation.

Third, most MOTAD applications use approximated standard errors as a measure of risk. When using such a measure, the risk aversion parameters can be interpreted as the number of standard errors one wishes to discount income. Coupling this with a normality assumption permits one to associate a confidence limit with the risk aversion parameter. For example, a risk aversion parameter equal to one means that level of income which occurs at one standard error below the mean is maximized. Assuming normality, this level of income is 84% sure to occur.

Fourth, one must have empirical values for the risk aversion parameter. All the E-V approaches are applicable to its discovery. The most common approach with MOTAD models has been based on observed behavior. The procedure has been to: a) take a vector of observed solution variables, (i.e. acreages); b) parameterize the risk aversion coefficient in small steps (e.g., 0.25) from 0 to 2.5, at each point computing a measure of the difference between the model solution and observed behavior; and c) select the risk aversion parameter value for which the smallest dispersion is found between the model solution values and the observed values (for examples see Hazell et al.; Brink and McCarl; Simmons and Pomareda; or Nieuwoudt, et al.).

Fifth, the MOTAD model does not have a general direct relationship to a theoretical utility function. Some authors have discovered special cases under which there is a link (see Johnson and Boeljhe(1981,1983) and their subsequent exchange with Buccola). Largely, the MOTAD model has been presented as an approximation to the E-V model. However, with the advances in nonlinear programming algorithms the approximation motivation is largely gone (McCarl and Onal), but MOTAD may have application to non-normal cases (Thomson and Hazell).

Sixth, McCarl and Bessler derive a link between the E-standard error and E-V risk aversion parameters as follows:

Consider the models

$$\begin{array}{llll} \text{Max} & cX - \Psi\ \sigma^2(X) & & \\ \text{s.t.} & AX & \leq & b \\ & X & \geq & 0 \end{array} \qquad \text{versus} \qquad \begin{array}{llll} \text{Max} & cX - \xi\ \sigma(X) & & \\ \text{s.t.} & AX & \leq & b \\ & X & \geq & 0 \end{array}$$

The first order conditions assuming X is nonzero are

$$c - 2\ \Psi\ \sigma(X)\frac{\partial\sigma(X)}{\partial X} - \lambda A = 0 \qquad\qquad c - \xi\ \frac{\partial\sigma(X)}{\partial X} - \lambda\ A = 0$$

For these two solutions to be identical in terms of X and λ, then

$$\Psi = \frac{\xi}{2\ \sigma(X)}$$

Thus, the E-V risk aversion coefficient will equal the E-standard error model risk aversion coefficient divided by twice the standard error. This explains why E-V risk aversion coefficients are usually very small (i.e., an E-standard error risk aversion coefficient usually ranges from 0 - 3 which implies when the standard error of income is $10,000 the E-V risk aversion coefficient range of 0 - .000015). Unfortunately, since ξ is a function of σ which depends on X, this condition must hold ex post and cannot be imposed a priori. However, one can develop an approximate a priori relationship between the risk aversion parameters given an estimate of the standard error.

The seventh and final comment regards model sensitivity. Schurle and Erven show that several plans with very different solutions can be feasible and close to the plans on the efficient frontier. Both results place doubt on strict adherence to the efficient frontier as a norm for decision making. (Actually the issue of near optimal solutions is much broader than just its role in risk models.) The MOTAD model has been rather widely used. Early uses were by Hazell (1971); Hazell and Scandizzo; Hazell et al. (1983); Simmons and Pomareda; and Nieuwoudt, et al. In the late 1970's the model saw much use. Articles from 1979 through the mid 1980s in just the American Journal of Agricultural Economics include Gebremeskel and Shumway; Schurle and Erven; Pomareda and Samayoa; Mapp, et al.; Apland, et al. (1980); and Jabara and Thompson. Boisvert and McCarl provide a recent review.

14.3.1.3 Toward A Unified Model

The E-V and MOTAD models evolved before many software developments. As a consequence, the models were formulated to be easily solved with 1960's and 70's software. A more extensive unified model formulation is possible today. The E-standard error form of this model is as follows

$$
\begin{array}{llllll}
\text{Max} & & \overline{Inc} - \Phi \left\{ \sum_k p_k \left[(d_k^+)^2 + (d_k^-)^2 \right] \right\}^{0.5} & & & \\
\text{s.t.} & \sum_j a_{ij} X_j & & & \leq b_i & \text{for all } i \\
& \sum_j c_{kj} X_j - Inc_k & & & = 0 & \text{for all } k \\
& \sum_k p_k Inc_k - \overline{Inc} & & & = 0 & \\
& Inc_k - \overline{Inc} - & d_k^+ + d_k^- & & = 0 & \text{for all } k \\
& X_j, & d_k^+, \quad d_k^- & & \geq 0 & \text{for all } j,k \\
& Inc_k, \quad \overline{Inc} & & & \lesseqgtr 0 & \text{for all } k
\end{array}
$$

In this model the resource constraints continue to appear. But we introduce a new variable (Inc_k) which is income under state of nature k. This is equated with income arising under the k^{th} state of nature. In turn, a variable is entered for average income ($\overline{Inc}$) which is equated to the probabilities (p_k) times the income levels. This variable appears in the objective function reflecting expected income maximization. Finally, deviations between the average and state of nature dependent income levels are treated in deviation constraints where d_k^+ indicates income above the average level whereas d_k^- indicates shortfalls. The objective function is then modified to include the probabilities and deviation variables. Several possible objective function formulations are possible. The objective function formulation above is E-standard error without approximation. Note that the term in parentheses contains the summed, probabilistically weighted, squared deviations from the mean and is by definition equal to the variance. In turn, the square root of this term is the standard deviation and Φ would be a risk aversion parameter which would range between zero and 2.5 in most circumstances (as explained in the MOTAD section).

This objective function can also be reformulated to be equivalent to either the MOTAD or E-V cases. Namely, in the E-V case if we drop the 0.5 exponent then the bracketed term is variance and the model would be E-V. Similarly, if we drop the 0.5 exponent and do not square the deviation variables then a MOTAD model arises.

This unifying framework shows how the various models are related and indicates that covariance is considered in any of the models. An example is not presented here although the files UNIFY, EV2 and MOTAD2 give GAMS implementations of the unified E-standard error, E-V and MOTAD versions. The resultant solutions are identical to the solution for E-V and MOTAD examples and are thus not discussed further.

14.3.1.4 Safety First

Roy posed a different approach to handling objective function uncertainty. This approach, the Safety First model, assumes that decision makers will choose plans to first assure a given safety level for income. The formulation arises as follows: assume the model income level under all k states of nature $\left(\sum c_{kj} X_j \right)$ must exceed the safety level (S). This can be assured by entering the constraints

$$\sum_j c_{kj} X_j \geq S \quad \text{for all } k$$

The overall problem then becomes

$$\begin{aligned} \text{Max} \quad & \sum_j \bar{c}_j X_j \\ \text{s.t.} \quad & \sum_j a_{ij} X_j \leq b_i \quad \text{for all } i \\ & \sum_j c_{kj} X_j \geq S \quad \text{for all } k \\ & X_j \geq 0 \quad \text{for all } j \end{aligned}$$

where S is the safety level.

14.3.1.4.1 Example

A formulation using the data from the E-V example and a safety level of S is given in Table 14.8 and a GAMS implementation is in the file SAFETY. This example was solved for safety levels ranging from -$100 to +$50. The solution (Table 14.9) at S = $100 gives the profit maximizing linear programming solution. As the safety level is increased the solutions reflect a diversification between X_3 and X_2. These solutions exhibit the same sort of behavior as in the previous examples. As the safety level increases a more diversified solution arises with an accompanying reduction in risk and a decrease in expected value. For example at S = $50 the mean has dropped from $148.00 to $135.00, but the standard error is cut by more than two-thirds.

14.3.1.4.2 Comments

The safety first model has not been extensively used empirically although Target MOTAD as discussed in the next section is a more frequently used extension. However, the Safety First model is popular as an analytical model in characterizing decision making. For a review and more extensive discussion see Barry.

14.3.1.5 Target MOTAD

The Target MOTAD formulation developed by Tauer, incorporates a safety level of income while also allowing negative deviations from that safety level. Given a target level of T, the formulation is

$$\begin{aligned}
\text{Max} \quad & \sum_j \bar{c}_j X_j \\
\text{s.t.} \quad & \sum_j a_{ij} X_j & & \leq b_i \quad \text{for all } i \\
& \sum_j c_{kj} X_j + \text{Dev}_k & & \geq T \quad \text{for all } k \\
& \sum_k p_k \text{Dev}_k & & \leq \lambda \\
& X_j, \quad \text{Dev}_k & & \geq 0 \quad \text{for all } j,k
\end{aligned}$$

All definitions are as above except p_k is the probability of the kth state of nature; T is the target income level (somewhat analogous to S in the safety first model); the variable Dev_k is the negative deviation of income, allowing income under the kth state of nature to fall below target income; and λ is the maximum average income shortfall permitted. The equation containing T gives the relationship between income under the kth state of nature and a target income level. The variable Dev_k is non-zero if the kth income result falls below T. The constraint with the right hand side of λ limits the average shortfall. Thus, the Target MOTAD model has two parameters relating to risk (T and λ) which must be specified. These, in turn, can be parameterized to yield different risk solutions.

14.3.1.5.1 Example

Using the data from the earlier examples and assuming each state of nature is equally probable (P_k = 1/10) yields the formulation given in Table 14.10 and the GAMS formulation is in the file TARGET.

The Target MOTAD example was solved with a safety level of $120.00 with the allowable deviation from the safety level varied from allowing as much as $120.00 average deviation to as little as $3.60. The solution behavior (Table 14.11) again largely mirrors that observed in the prior examples. Namely, when a large deviation is allowed, the profit maximizing solution is found, but as the allowable deviation gets smaller, then X_2 enters and then finally X_1. Again a sacrifice in expected income yields less risk.

14.3.1.5.2 Comments

Target MOTAD has not been applied as widely as other risk programming models. However, it is consistent with second degree stochastic dominance (Tauer). Use of Target MOTAD requires specification of two parameters, T and λ. No attempt has been made to determine consistency between a T, λ choice and the Arrow-Pratt measure of risk aversion. Nor is there theory on how to specify T and λ. The target MOTAD and original MOTAD models can be related. If one makes λ a variable with a cost in the objective function and makes the target level a variable equal to expected income, this becomes the MOTAD model.

Another thing worth noting is that the set of Target MOTAD solutions are continuous so that there is an infinite number of solutions. In the example, any target deviation between $24.00 and $12.00 would be a unique solution and would be a convex combination of the two tabled solutions.

McCamley and Kliebenstein outline a strategy for generating all target MOTAD solutions, but it is still impossible to relate these solutions to more conventional measures of risk preferences.

Target MOTAD has been used in a number of contexts. Zimet and Spreen formulate a farm production implementation while Curtis et al., and Frank et al., studied crop marketing problems.

14.3.1.6 DEMP

Lambert and McCarl (1985) introduced the Direct Expected Maximizing Nonlinear Programming (DEMP) formulation, which maximizes the expected utility of wealth. DEMP was designed as an alternative to E-V analysis, relaxing some of the restrictions regarding the underlying utility function. The basic DEMP formulation requires specification of a utility of wealth function U(W), a level of initial wealth (W_o), and the probability distribution of the objective function parameters (C_{kj}). The basic formulation is

$$
\begin{array}{lrcll}
\text{Max} & \sum_k p_k \, U(W_k) & & & \\
\text{s.t.} & \sum_j a_{ij} X_j & \leq & b_i & \text{for all } i \\
 & W_k - \sum_j c_{kj} X_j & = & W_o & \text{for all } k \\
 & W_k & \lesseqgtr & 0 & \text{for all } k \\
 & X_j & \geq & 0 & \text{for all } j
\end{array}
$$

where p_k is the probability of the k^{th} state of nature;

W_o is initial wealth;

W_k is the wealth under the k th state of nature; and

c_{kj} is the return to one unit of the j th activity under the k th state of nature.

14.3.1.6.1 Example

Suppose an individual has the utility function for wealth of the form $U = (W)^{power}$ with an initial wealth (W_0) of 100, and is confronted with the decision problem data as used in the E-V example. The relevant DEMP formulation appears in Table 14.12 with the solution for varying values of the exponent appearing in Table 14.13. The GAMS formulation is called DEMP.

The example model was solved for different values of the exponent (power). The exponent was varied from 0.3 to 0.0001. As this was varied, the solution again transitioned out of sole reliance on stock three into reliance on stocks two and three. During the model calculations, transformations were done on the shadow price to convert it into dollars. Following Lambert and McCarl, this may be converted into an approximate value in dollar space by dividing by the marginal utility of average income i.e., dividing the shadow prices by the factor.

$$\mu^* = \mu \; / \; \frac{\partial \; U(\overline{W})}{\partial W}$$

Preckel, Featherstone, and Baker discuss a variant of this procedure.

14.3.1.6.2 Comments

The DEMP model has two important parts. First, note that the constraints involving wealth can be rearranged to yield

$$W_k = W_o + \sum_j c_{kj} X_j$$

This sets wealth under the k th state of nature equal to initial wealth plus the increment to wealth due to the choice of the decision variables.

Second, note that the objective function equals expected utility. Thus the formulation maximizes expected utility using the empirical distribution of risk without any distributional form assumptions and an explicit, exact specification of the utility function.

Kaylen, et al., employ a variation of DEMP where the probability distributions are of a known continuous form and numerical integration is used in the solution. The DEMP model has been used by Lambert and McCarl(1989); Lambert; and Featherstone et al.

Yassour, et al., present a related expected utility maximizing model called EUMGF, which embodies both an exponential utility function and distributional assumptions. They recognize that the maximization of expected utility under an exponential utility function is equivalent to maximization of the moment generating function (Hogg and Craig) for a particular probability distribution assumption. Moment generating functions have been developed analytically for a number of distributions, including the Binomial, Chi Square, Gamma, Normal and Poisson distributions. Collender and Zilberman and Moffit et al. have applied the EUMGF model. Collender and Chalfant have proposed a version of the model no longer requiring that the form of the probability distribution be known.

14.3.1.7 Other Formulations

The formulations mentioned above are the principal objective function risk formulations which have been used in applied mathematical programming risk research. However, a number of other formulations have been proposed. Alternative portfolio models such as those by Sharpe; Chen and Baker; Thomas et al.(1972) exist. Other concepts of target income have also been pursued (Boussard and Petit) as have models based upon game theory concepts (McInerney [1967, 1969]; Hazell and How; Kawaguchi and Maruyama; Hazell(1970); Agrawal and Heady; Maruyama; and Low) and Gini coefficients (Yitzhaki). These have all experienced very limited use and are therefore not covered herein.

14.3.2 Right Hand Side Risk

Risk may also occur within the right hand side (RHS) parameters. The most often used approach to RHS risk in a nonrecourse setting is chance-constrained programming. However, Paris(1979) has tried to introduce an alternative.

14.3.2.1 Chance Constrained Programming

The chance-constrained formulation was introduced by Charnes and Cooper and deals with uncertain RHS's assuming the decision maker is willing to make a probabilistic statement about the frequency with which constraints need to be satisfied. Namely, the probability of a constraint being satisfied is greater than or equal to a prespecified value α.

$$P\left(\sum_j a_{ij} X_j \le b_i\right) \ge \alpha$$

If the average value of the RHS ($\bar{b}_i$) is subtracted from both sides of the inequality and in turn both sides are divided by the standard deviation of the RHS (σ_{b_i}) then the constraint becomes

$$P\left[\frac{\sum_j a_{ij} X_j - \bar{b}_i}{\sigma_{b_i}} \le \frac{(b_i - \bar{b}_i)}{\sigma_{b_i}}\right] \ge \alpha$$

Those familiar with probability theory will note that the term

$$\frac{(b_i - \bar{b}_i)}{\sigma_{b_i}}$$

gives the number of standard errors that b_i is away from the mean. Let Z denote this term.

When a particular probability limit (α) is used, then the appropriate value of Z is Z_α and the constraint becomes

$$P\left[\frac{\sum_j a_{ij} X_j - \bar{b}_i}{\sigma_{b_i}} \le Z_\alpha\right] \ge \alpha$$

Assuming we discount for risk, then the constraint can be restated as

$$\sum_j a_{ij} X_j \le \bar{b}_i - Z_\alpha \sigma_{b_i}$$

which states that resource use ($\sum a_{ij}X_j$) must be less than or equal to average resource availability less the standard deviation times a critical value which arises from the probability level.

Values of Z_α may be determined in two ways: a) by making assumptions about the form of the probability distribution of b_i (for example, assuming normality and using values for the lower tail from a standard normal probability table); or b) by relying on the conservative estimates generated by using Chebyshev's inequality, which states the probability of an estimate falling greater than M standard deviations away from the mean is less than or equal to one divided by M^2. Using the Chebyshev inequality one needs to solve for that value of M such that $(1-\alpha)$ equals $1/M^2$. Thus, given a probability α, the Chebyshev value of Z_α is given by the equation $Z_\alpha=(1-\alpha)^{-0.5}$. Following these approaches, if one wished an 87.5 percent probability, a normality assumption would discount 1.14 standard deviations and an application of the Chebyshev inequality would lead to a discount of 2.83 standard deviations. However, one should note that the Chebyshev bound is often too large.

14.3.2.1.1 Example

The example problem adopted for this analysis is in the context of the resource allocation problem from Chapter V. Here three of the four right hand sides in that problem are presumed to be stochastic with the distribution as given in Table 14.14. Treating each of these right hand side observations as equally likely, the mean value equals those numbers that were used in the resource allocation problem and their standard errors respectively are as given in Table 14.14. Then the resultant chance constrained formulation is

$$\begin{array}{llllllllllllll}
\text{Max} & 67X_1 & + & 66X_2 & + & 66.3X_3 & + & 80X_4 & + & 78.5X_5 & + & 78.4X_6 & & \\
\text{s.t} & 0.8X_1 & + & 1.3X_2 & + & 0.2X_3 & + & 1.2X_4 & + & 1.7X_5 & + & 0.5X_6 & \leq & 140 - 9.63\ Z_\alpha \\
& 0.5X_1 & + & 0.2X_2 & + & 1.3X_3 & + & 0.7X_4 & + & 0.3X_5 & + & 1.5X_6 & \leq & 90 - 3.69\ Z_\alpha \\
& 0.4X_1 & + & 0.4X_2 & + & 0.4X_3 & + & X_4 & + & X_5 & + & X_6 & \leq & 120 - 8.00\ Z_\alpha \\
& X_1 & + & 1.05X_2 & + & 1.1X_3 & + & 0.8X_4 & + & 0.82X_5 & + & 0.84X_6 & \leq & 125
\end{array}$$

The GAMS implementation is the file CHANCE. The solutions to this model were run for Z values corresponding to 0, 90 ,95, and 99 percent confidence intervals under a normality assumption. The right hand sides and resultant solutions are tabled in Table 14.15. Notice as the Z_α value is increased, then the value of the uncertain right hand side decreases. In turn, production decreases as does profit. The chance constrained model discounts the resources available, so one is more certain that the constraint will be met. The formulation also shows how to handle simultaneous constraints. Namely the constraints may be treated individually. Note however this requires an assumption that the right hand sides are completely dependent. The results also show that there is a chance of the constraints being exceeded but no adjustment is made for what happens under that circumstance.

14.3.2.1.2 Comments

Despite the fact that chance constrained programming (CCP) is a well known technique and has been applied to agriculture (e.g., Boisvert, 1976; Boisvert and Jensen, 1973; and Danok et al., 1980) and water management (e.g., Eisel; Loucks; and Maji and Heady) its use has been limited and controversial. See the dialogue in Blau; Hogan, et al.; and Charnes and Cooper (1959).

The major advantage of CCP is its simplicity; it leads to an equivalent programming problem of about the same size and the only additional data requirements are the standard errors of the right hand side. However, its only decision theoretic underpinning is Simon's principle of satisficing (Pfaffenberger and Walker).

This CCP formulation applies when either one element of the right hand side vector is random or when the distribution of multiple elements is assumed to be perfectly correlated. The procedure has been generalized to other forms of jointly distributed RHS's by Wagner (1975). A fundamental problem with chance constrained programming (CCP) is that it does not indicate what to do if the recommended solution is not feasible. From this perspective, Hogan et al., (1981), conclude that "... there is little evidence that CCP is used with the care that is necessary" (p. 698) and assert that recourse formulations should be used.

14.3.2.2 A Quadratic Programming Approach

Paris(1979) proposed a quadratic programming model which permits RHS risk in an E-V context. In contrast to chance constrained programming, the formulation treats inter-dependencies between the RHS's. The formulation is developed through an application of non-linear duality theory and is

$$\begin{aligned} \text{Max} \quad & \bar{c}X - \phi X'S_cX - \Theta Y'S_bY \\ \text{s.t.} \quad & AX - \Theta S_bY \le \bar{b} \\ & X \ , \ Y \ge 0 \end{aligned}$$

where X is the vector of activities; ϕ and Θ are risk aversion coefficients with respect to variance in returns and the RHS. S_c and S_b are variance-covariance matrices of returns and the RHS's, respectively; Y is the vector of dual variables, A is the matrix of technical coefficients, and $\bar{b}$ is the vector of expected values of the RHS's.

This primal model explicitly contains the dual variables and the variance-covariance matrix of the RHS's. However, the solutions are not what one might expect. Namely, in our experience, as right hand side risk aversion increases, so does expected income. The reason lies in the duality implications of the formulation. Risk aversion affects the dual problem by making its objective function worse. Since the dual objective function value is always greater than the primal, a worsening of the dual objective via risk aversion improves the primal. A manifestation of this appears in the way the risk terms enter the constraints. Note given positive Θ and S_b, then the sum involving Θ and Y on the left hand side augments the availability of the resources. Thus, under any nonzero selection of the dual variables, as the risk aversion parameter increases so does the implicit supplies of resources. Dubman et al., and Paris(1989) debate these issues, but the basic flaw in the formulation is not fixed. Thus we do not recommend use of this formulation and do not include an example.

14.3.3 Technical Coefficient Risk

Risk can also appear within the matrix of technical coefficients. Resolution of technical coefficient uncertainty in a non-recourse setting has been investigated through two approaches. These involve an E-V like procedure (Merrill), and one similar to MOTAD (Wicks and Guise).

14.3.3.1 Merrill's Approach

Merrill formulated a nonlinear programming problem including the mean and variance of the risky a_{ij}'s into the constraint matrix. Namely, one may write the mean of the risky part as $\sum \overline{a_{ij}} X_j$ and its variance as $\sum\sum X_j X_n \sigma_{inj}$ where $\overline{a_{ij}}$ is the mean value of the a_{ij}'s and σ_{inj} is the covariance of the a_{ij} coefficients for activities n and j in row i. Thus, a constraint containing uncertain coefficients can be rewritten as

$$\sum_j \overline{a_{ij}} X_j + \phi \sum_j \sum_n X_j X_n \sigma_{inj} \leq b_i \quad \text{for all } i$$

or, using standard deviation,

$$\sum_j a_{ij} X_j + \phi \left(\sum_j \sum_k X_j X_n \sigma_{inj} \right)^{0.5} \leq b_i \quad \text{for all } i$$

Note that the term involving σ_{inj} is added inflating resource use above the average to reflect variability, thus a safety cushion is introduced between average resource use and the reserve limit. The parameter Φ determines the amount of safety cushion to be specified exogenously and could be done using distributional assumptions (such as normality) or Chebyshev's inequality as argued in McCarl and Bessler. The problem in this form requires usage of nonlinear programming techniques.

Merrill's approach has been unused largely since it was developed at a time when it was incompatible with available software. However, the MINOS algorithm in GAMS provides capabilities for handling the nonlinear constraint terms (although solution times may be long -- McCarl and Onal). Nevertheless the simpler Wicks and Guise approach discussed below is more likely to be used. Thus no example is given.

14.3.3.2 Wicks and Guise Approach

Wicks and Guise provided a LP version of an uncertain a_{ij} formulation based on Hazell(1971) and Merrill's models. Specifically, given that the i^{th} constraint contains uncertain a_{ij}'s, the following constraints may be set up.

$$\begin{aligned}
\sum_j a_{ij} X_j & & & + \phi D_i & \leq b_i & \\
\sum_j (a_{kij} - \overline{a_{ij}}) X_j & - d_{ki}^{+} & + d_{ki}^{-} & & = 0 & \quad \text{for all } k \\
& \sum_k (d_{ki}^{+} & + d_{ki}^{-}) & - D_i & = 0 &
\end{aligned}$$

Here the first equation relates the mean value of uncertain resource usage plus a risk term (ϕD_i) to the right hand side, while the second computes the deviation ($a_{kij} - \bar{a}_{ij}$) incurred from the k th joint observation on all a_{ij}'s and sums it into a pair of deviation variables (d^+_{ki} , d^-_{ki}). These deviation variables are in turn summed into a measure of total absolute deviation (D_i) by the third equation. The term ϕD_i then gives the risk adjustment to the mean resource use in constraint i where ϕ is a coefficient of risk aversion.

The Wicks and Guise formulation is essentially this; however, Wicks and Guise convert the total absolute deviation into an estimate of standard deviation using a variant of the Fisher constant but we will use the one discussed above

$$\Delta D - \sigma = 0$$

where $\Delta = (\pi/(2n(n-1)))^{.5}$ and σ is the standard error approximation. The general Wicks Guise formulation is

$$
\begin{array}{llllllll}
\text{Max} & \sum_j c_j X_j & & & & & & \\
\text{s.t.} & \sum_j \bar{a}_{ij} X_j & & & + \phi\sigma_i & \leq & b_i & \text{for all } i \\
& \sum_j (a_{kij} - \bar{a}_{ij}) X_j & - d^+_{ki} + d^-_{ki} & & & = & 0 & \text{for all } i,k \\
& & \sum_k (d^+_{ki} + d^-_{ki}) & - D_i & & = & 0 & \text{for all } i \\
& & & \Delta D_i & - \sigma_i & = & 0 & \text{for all } i \\
& X_j, & d^+_{ki}, \quad d^-_{ki}, & D_i, & \sigma_i & \geq & 0 & \text{for all } j,k,i
\end{array}
$$

14.3.3.2.1 Example

Suppose we introduce ingredient uncertainty in the context of the feed problem as discussed in Chapter V. Suppose one is using three feed ingredients corn, soybeans, and wheat while having to meet energy and protein requirements. However, suppose that there are four states of nature for energy and protein nutrient content as given in Table 14.16. Assume that the unit price of corn is 3 cents, soybeans 6 cents, and wheat 4 cents and that the energy requirements are 80% of the unit weight of the feed while the protein requirement is 50%. In turn, the GAMS formulation of this is called WICKGUIS and a tableau is given in

Table 14.17.

The solution to the Wicks Guise example model are given in Table 14.18. Notice in this table when the risk aversion parameter is 0 then the model feeds corn and wheat, but as the risk aversion parameter increases the model first reduces its reliance on corn and increases wheat, but as the risk aversion parameter gets larger and larger one begins to see soybeans come into the answer. Notice across these solutions, risk aversion generally increases the average amount of protein with reductions in protein variability. As the risk aversion parameter increases, the probability of meeting the constraint increases. Also notice that the shadow price on protein monotonically increases indicating that it is the risky ingredient driving the model adjustments. Meanwhile average energy decreases, as does energy variation and the shadow price on energy is zero, indicating there is sufficient energy in all solutions.

14.3.3.2.2 Comments

The reader should note that the deviation variables do not work well unless the constraint including the risk adjustment is binding. However, if it is not binding, then the uncertainty does not matter.

The Wicks and Guise formulation has not been widely used. Other than the initial application by Wicks and Guise the only other application we know of is that of Tice.

Several other efforts have been made regarding a_{ij} uncertainty. The method used in Townsley and later by Chen (1973) involves bringing a single uncertain constraint into the objective function. The method used in Rahman and Bender involves developing an over-estimate of variance.

14.3.4 Multiple Sources of Risk

Many problems have C's, A's and b's which are simultaneously uncertain. The formulations above may be combined to handle such a case. Thus, one could have a E-V model with several constraints handled via the Wicks Guise and/or chance constrained techniques. There are also techniques for handling multiple sources of risk under the stochastic programming with recourse topic.

14.4 Sequential Risk-Stochastic Programming with Recourse

Sequential risk arises as part of the risk as time goes on and adaptive decisions are made. Consider the way that weather and field working time risks are resolved in crop farming. Early on, planting and harvesting weather are uncertain. After the planting season, the planting decisions have been made and the planting weather has become known, but harvesting weather is still uncertain. Under such circumstances a decision maker would adjust to conform to the planting pattern but would still need to make harvesting decisions in the face of harvest time uncertainty. Thus sequential risk models must depict adaptive decisions along with fixity of earlier decisions (a decision maker cannot always undo earlier decisions such as planted acreage). Nonsequential risk, on the other hand, implies that a decision maker chooses a decision now and finds out about all sources of risk later.

All the models above are nonsequential risk models. Stochastic programming with recourse (SPR) models are used to depict sequential risk. The first of the models was originally developed as the "two-stage" LP formulation by Dantzig (1955). Later, Cocks devised a model with N stages, calling it discrete stochastic programming. Over time, the whole area has been called stochastic programming with recourse (SPR). We adopt this name.

14.4.1 Two stage SPR formulation

Suppose we set up a two stage SPR formulation. Such formulations contain a probability tree (Figure 14.3). The nodes of the tree represent decision points. The branches of the tree represent alternative possible states. A two stage model has one node and set of decision variables (X) at the first stage, with the second stage containing branches associated with the resolved uncertainty from the first stage and associated decision nodes (Z_k).

Suppose the variables X_j indicate the amount of the j^{th} alternative which is employed in the first stage. There is an associated set of resource constraints where the per unit usage of the i^{th} resource by X_j is a_{ij} and the endowment of the resources b_i. Suppose that the outcome of X_j is uncertain and dependent on state of nature where the quantity of the m^{th} output item produced is d_{mjk} where k designates state of nature. Let us also define c_j as the objective function coefficient for X_j. In the second stage, the variables are Z_{nk}, where n represents the n^{th} alternative for production and k identifies state of nature. Here we have different decision variables for each second stage state of nature. For example, we have the amount of stock sold if the market has been moving up and the amount of stock sold if the market is moving down, with second stage decisions that depend upon the resultant state of nature after the first stage. We also have parameters which give the amount of the m^{th} output item carrying over from stage one (f_{mnk}) while g_{wnk} gives the amount of the w^{th} resource utilized by Z_{nk}. Finally, the objective function parameter for Z_{nk} is e_{nk}. The model also requires definition of right hand side parameters where s_{wk} is the amount of the w^{th} resource available under the k^{th} state of nature. In setting this model up we also define a set of accounting variables Y_k, which add up income under the states of nature. Finally suppose p_k gives the probability of state k. The composite model formulation is

$$
\begin{array}{llllllll}
\text{Max} & \sum_k p_k Y_k & & & & & & \\
\text{s.t.} & -Y_k & + \sum_j c_j X_j & + \sum_n e_{nk} Z_{nk} & = & 0 & & \text{for all } k \\
 & & \sum_j a_{ij} X_j & & \leq & b_i & & \text{for all } i \\
 & & -\sum_j d_{mjk} X_j & + \sum_n f_{mnk} Z_{nk} & \leq & 0 & & \text{for all } m,k \\
 & & & \sum_n g_{wnk} Z_{nk} & \leq & s_{wk} & & \text{for all } w,k \\
 & Y_k & & & \gtrless & 0 & & \text{for all } k \\
 & & X_j, & Z_{nk} & \geq & 0 & & \text{for all } j,n,k
\end{array}
$$

In this problem we have income variables for each of the k states of nature (Y_k) which are unrestricted in sign. Given that p_k is the probability of the k^{th} state of nature, then the model maximizes expected income. Note the income variable under the k^{th} state of nature is equated to the sum of the nonstochastic income from the first stage variables plus the second stage state of nature dependent profit contribution. Also note that since Z has taken on the subscript k, the decision variable value will in general vary by state of nature.

Several points should be noted about this formulation. First, let us note what is risky. In the second stage the resource endowment (S_{wk}), constraint coefficients (d_{mjk}, f_{mnk}, g_{wnk}) and objective function parameters (e_{nk}) are dependent upon the state. Thus, all types of coefficients (RHS, OBJ and A_{ij}) are potentially risky and their values depend upon the path through the decision tree.

Second, this model reflects a different uncertainty assumption for X and Z. Note Z is chosen with knowledge of the stochastic outcomes; however, X is chosen a priori, with it's value fixed regardless of the stochastic outcomes. Also notice that the first, third, and fourth constraints involve uncertain parameters and are repeated for each of the states of nature. This problem then has a single X solution and a Z solution for each state of nature. Thus, adaptive decision making is modeled as the Z variables are set conditional on the state of nature. Note that irreversabilities and fixity of initial decisions is modeled. The X variables are fixed across all second stage states of nature, but the Z variables adapt to the state of nature.

Third, let us examine the linkages between the stages. The coefficients reflect a potentially risky link between the predecessor (X) and successor (Z) activities through the third constraint. Note the link is essential since if the activities are not linked, then the problem is not a sequential decision problem. These links may involve the weighted sum of a number of predecessor and successor variables (i.e., an uncertain quantity of lumber harvested via several cutting schemes linked with use in several products). Also, multiple links may be present (i.e., there may be numerous types of lumber). The subscript m defines these links. A fourth comment relates to the nature of uncertainty resolution. The formulation places all uncertainty into the objective function, which maximizes expected income.

14.4.1.1 Example

Let us consider a simple farm planning problem. Suppose we can raise corn and wheat on a 100 acre farm. Suppose per acre planting cost for corn is $100 while wheat costs $60. However, suppose crop yields, harvest time requirements per unit of yield, harvest time availability and crop prices are uncertain. The deterministic problem is formulated as in Table 14.20 and file SPREXAM1. Here the harvest activities are expressed on a per unit yield basis and the income variable equals sales revenue minus production costs.

The uncertainty in the problem is assumed to fall into two states of nature and is expressed in Table 14.19. These data give a joint distribution of all the uncertain parameters. Here RHS's, a_{ij}'s and objective function coefficient's are uncertain.

Solution of the Table 14.20 LP formulation under each of the states of nature gives two very different answers. Namely under the first state of nature all acreage is in corn while under the second state of nature all production is in wheat. These are clearly not robust solutions.

The SPR formulation of this example is given in Table 14.21. This tableau contains one set of first stage variables (i.e., one set of corn growing and wheat growing activities) coupled with two sets of second stage variables after the uncertainty is resolved (i.e., there are income, harvest corn, and harvest wheat variables for both states of nature). Further, there is a single unifying objective function and land constraint, but two sets of constraints for the states of nature (i.e., two sets of corn and wheat yield balances, harvesting hour constraints and income constraints). Notice underneath the first stage corn and wheat production variables, that there are coefficients in both the state of nature dependent constraints reflecting the different uncertain yields from the first stage (i.e., corn yields 100 bushels under the first state of nature and 105 under the second; while wheat yields 40 under the first and 38 under the second). However, in the second stage resource usage for harvesting is independent. Thus, the 122 hours available under the first state of nature cannot be utilized by any of the activities under the second state of nature. Also, the crop prices under the harvest activities vary by state of nature as do the harvest time resource usages.

The example model then reflects, for example, if one acre of corn is grown that 100 bushels will be available for harvesting under state of nature one, while 105 will be available under state of nature two. In the optimum solution there are two harvesting solutions, but one production solution. Thus, we model irreversibility (i.e., the corn and wheat growing variable levels maximize expected income across the states of nature, but the harvesting variable levels depend on state of nature).

The SPR solution to this example is shown in Table 14.22. Here the acreage is basically split 50-50 between corn and wheat, but harvesting differs with almost 4900 bushels of corn harvested under the first state, where as 5100 bushels of corn are harvested under the second. This shows adaptive decision making with the harvest decision conditional on state of nature. The model also shows different income levels by state of nature with $21,059 made under state of nature one and $21,762 under state of nature two. Furthermore, note that the shadow prices are the marginal values of the resources times the probability of the state of nature. Thus, wheat is worth $3.00 under the first state of nature but taking into account that the probability of the first state of nature is 60% we divide the $3.00 by .6 we get the original $5.00 price. This shows the shadow prices give the contribution to the average objective function. If one wishes shadow prices relevant to income under a state of nature then one needs to divide by the appropriate probability.

The income accounting feature also merits discussion. Note that the full cost of growing corn is accounted for under both the first and second states of nature. However, since income under the first state of nature is multiplied by .6 and income under the second state of nature is multiplied by .4, then no double counting is present.

14.4.2 Incorporating Risk Aversion

The two stage model as presented above is risk neutral. This two stage formulation can be altered to incorporate risk aversion by adding two new sets of constraints and three sets of variables following the method used in the unified model above. An EV formulation is

$$
\begin{array}{lllllll}
\text{Max} & \bar{Y} & & - \psi \sum_k p_k (d_k^+ + d_k^-)^2 & & & \\
\text{s.t.} & -\bar{Y} & + \sum_k p_k Y_k & & = & 0 & \\
 & -\bar{Y} & + Y_k & - d_k^+ + d_k^- & = & 0 & \text{for all } k \\
 & + \sum_j c_j X_j & - Y_k & + \sum_n e_{nk} Z_{nk} & = & 0 & \text{for all } k \\
 & \sum_j a_{ij} X_j & & & \leq & b_i & \text{for all } i \\
 & - \sum_j d_{mjk} X_j & & + \sum_n f_{mnk} Z_{nk} & \leq & 0 & \text{for all } m,k \\
 & & & \sum_n g_{wnk} Z_{nk} & \leq & s_{wk} & \text{for all } w,k \\
 & \bar{Y}, & Y_k & & \lessgtr & 0 & \text{for all } k \\
 & X_j, & & d_k^+, \quad d_k^-, \quad Z_{nk} & \geq & 0 & \text{for all } j,n,k
\end{array}
$$

Note that within this formulation the first new constraint that we add simply accounts expected income into a variable $\bar{Y}$, while the second constraint computes deviations from expected income into new deviation variables d_k^+, d_k^- which are defined by state of nature. Further, the objective function is modified so it contains expected income minus a risk aversion parameter times the probabilistically weighted squared deviations (i.e., variance). This is as an EV model. The model may also be formulated in the fashion of the unified model discussed earlier to yield either a MOTAD or an E-standard deviation model.

14.4.2.1 Example

Suppose we use the data from the above Wicks Guise example but also allow decision makers once they discover the state of nature, to supplement the diet. In this case, suppose the diet supplement to correct for excess protein deviation costs the firm $0.50 per protein unit while insufficient protein costs $1.50 per unit. Similarly, suppose excess energy costs $1.00 per unit while insufficient energy costs $0.10. The resultant SPR tableau, portraying just two of the four states of nature included in the tableau, is shown in Table 14.23 (This smaller portrayal is only done to preserve readability, the full problem is solved). Notice we again have the standard structure of an SPR. Namely the corn, soybeans, and wheat activities are first stage activities, then in the second stage there are positive and negative nutrient deviations for each state as well as state dependent objective function and deviation variable accounting. Notice the average cost row adds the probabilistically weighted sums of the state of nature dependent variables into average cost while the cost deviation rows compute deviation under a particular state of nature. In turn, these deviations are weighted by the probability times the risk aversion parameter and are entered in the objective function. The deviation variables could be treated to form an E-V, MOTAD or E-Standard error formulation as in the unified model above. An E-standard deviation model will be used here and is implemented in the GAMS file FEEDSPR. Also note these activities repeat for the second state of nature and also would for the third and fourth if they were portrayed here.

The risk neutral solution to this problem is given in Table 14.24. Two solution aspects are worth discussing. First, notice that the first stage solution is to buy .283 pounds of corn, .362 pounds of soybeans, .355 pounds of wheat at an average cost of 6.7 cents. Cost varies across the states of nature with cost under the first state equaling 8.1 cents, while under the second state it is 8.3, 5.2 under the third state and 5.1 under the fourth state. The cost variation arises as the protein and energy shortfall and excess variables take on different values in order to mitigate nutrient fluctuation.

The model was also solved for risk aversion. The results in Table 14.25 show the solutions from the example model under varying risk aversion coefficients for a standard deviation implementation. Table 14.25 gives the changes in corn, soybean, and wheat usage, as well as average income and standard error of income as the risk aversion parameter is changed for an E-standard deviation formulation as implemented in the file FEEDSPR. Here the risk aversion parameter was varied from 0.0 up to 0.6. As risk aversion increases the average cost of the diet increases, but the standard error of the cost of the diet falls with cost variation between the various states of nature narrowing. Namely under risk neutrality cost ranges from 5.1 cents to 8.1 cents with a standard error of 1.5 cents, however by the time the risk aversion parameter is up to .4 the cost varies from only 6.7 to 7.4 cents with a standard error of two tenths of a cent, at the expense of a 0.4 cent increase in average diet cost. Thus, as risk aversion increases, the model adopts a plan which stabilizes income across all of the states of nature.

14.4.3 Extending to Multiple Stages

The models above are two stage models with a set of predecessor activities followed by sets of successor activities for each state of nature. It is possible to formulate a multiple stage model as done by Cocks. In such a model however, it is relatively cumbersome to express a general formulation. Thus, we will express this model only in terms of an example (See Cocks for an N stage formulation and Boisvert and McCarl for a three stage one). Let us formulate a relatively simple stock model. Suppose that a firm starts with an initial inventory 100 units of common stock and is trying to maximize average ending worth. In doing this, suppose that the stock can be sold in one of three time periods. The first one which is nonstochastic, the second one which is characterized by two states of nature, and the third which is characterized by two additional states of nature. In describing the states of nature the following data are relevant. In period one (today) the firm knows the price is $2.00. In period two, the firm is uncertain of the interest rate between periods and the future price. Assume that under state of nature 1, the interest rate between period one and two for any stock sold is one percent while it is two percent under the second state of nature 2. Simultaneously the stock price is $2.20 under the first state of nature and $2.05 under the second. Going into the third state of nature, the interest rate is conditional on which state of nature was drawn for the second state. Thus, in the third stage if the first state arose the third stage interest rates are then either 6% (A) or 4% (B). On the other hand if the second state occurs, the interest rate will either be 7% (A) or 3% (B). Third stage crop prices are dependent of which of the two third stage states of nature occur. Under the first state of nature (A) the price is $2.18, while under the second one it is $2.44. The third stage probabilities are also conditional. Namely, after the first stage one gets state 1 occurring 70% of the time while state 2 occurs 30%of the time. When state 2 results out of stage one then the third stage probability for state A is 60% and is 40% for state B. On the other hand, these probabilities change to .7 and .3 if the second state happened out of stage 1.

The resultant formulation of this problem is given in Table 14.26 and file SELLSPR. Here, again, there is one set of period one variables which refer to either keeping or selling the stock; two sets of period two variables, which refer again to keep or sell the stock under each second stage state of nature; and four sets of period three variables for selling the stock and accounting ending net worth under all the third stage states of nature. Note in the first period, if the stock is kept, it carries over from the first period to both states of nature in the second stage. Then in the second period the keep activity from the first period provides stock that could either be sold or kept on into the third. In turn, if stock is kept in the second stage, it is held over to both third period states of nature which follow that second period state of nature. Notice the probabilities of each of the final states are reflected in the average ending worth. The worth under period three state A following period two state one is multiplied 0.42 which reflects the 70% probability of period two state one times the 60% conditional probability of period three state A. Also, notice the prices as they enter the ending worth by state of nature are the sales price in the relevant period times 1 plus interest earned in the interim periods. Thus, the ending worth of period one sales following period two state one and period three state A is 2.1412. This reflects the original sales price of $2.00, the 1% interest into the second period and the 6% interest into the third period. The solution to this model is given in Table 14.27.

14.4.4 Model Discussion

The SPR model is perhaps the most satisfying of the risk models. Conceptually it incorporates all sources of uncertainty: right hand side, objective function and technical coefficients while allowing adaptive decisions. However, the formulations suffer from the "curse of dimensionality." Each possible final state of nature leads to another set of stage two or later activities and large models can result from relatively simple problems. For example, consider having ten values of two right hand sides which were independently distributed. This would lead to 100 terminal states or sets of rows. However, such models can be computationally tractable, since the sparsity and repeated structure tend to make such problems easier to solve than their size would imply. Thus, one of the things to be cautious about when using this particular formulation is size. When dealing with such a model, it is often advisable to determine the critical sources of uncertainty which should be extensively modeled. Uncertainties other than the "most critical" may be handled with such methods as MOTAD, Wicks and Guise, or chance-constrained as discussed above. Sources of uncertainty which are not important in the problem may be held at their expected values (see Tice for an example). Thus, with careful application, this type of model can be quite useful.

Agricultural economics applications include Yaron and Horowit (1972a); Garoian, et al.; Apland, et al.(1981); Lambert and McCarl(1989); Leatham and Baker; McCarl and Parandvash; and the early papers by Rae (1971a, 1971b). Hansotia; Boisvert and McCarl; and Apland and Kaiser provide literature reviews.

14.5 General Comments On Modeling Uncertainty

As demonstrated above, there are a number of ways of handling uncertainty when modeling. Several aspects of these types of models need to be pointed out. First, all the formulations convert the problems to a deterministic equivalent. Basically, it is assumed that the decision maker is certain of the risk and reacts to it optimally by discounting the objective function, $a_{ij's}$ or right hand sides. Obviously this means the modeler must assume knowledge of the distribution of risk faced by a decision maker and the risk aversion coefficient.

The second set of comments regards data. Important parameters within the context of risk models are the expectation of the coefficient value and its probability distribution around that expectation. The most common practice for specification of these parameters is to use the historical mean and variance. This, however, is neither necessary nor always desirable. Fundamentally, the measures that are needed are the value expected for each uncertain parameter and the perceived probability distribution of deviations from that expectation (with joint distributions among the various uncertain parameters). The parameter expectation is not always a historical mean. This is most unrealistic in cases where there has been a strong historical trend (as pointed out by Chen, 1971). There is a large body of literature dealing with expectations and/or time series analysis (see Judge for an introduction), and some use of these results and procedures appears desirable.

Data are most often generated historically; however, observations could be generated by several other means. For example, observations could be developed from a simulation model (see Dillon, et al.), from a forecasting equation (see Lambert and McCarl(1989)), or from subjective interrogation of the decision maker (see Sri Ramaratnam et al.). There are cases where these other methods are more appropriate than history due to such factors as limited historical data (say, on the price of a new product) or major structural changes in markets. Naturally, the form in which the data are collected depends on the particular application involved.

A final comment on data regards their probabilistic nature. Basically when using historically based means and variance one is assuming that all observations are equally probable. When this assumption is invalid, the model is modified so that the value expected is the probabilistically weighted mean (if desired) and the variance formula includes the consideration of probability (see Anderson, et al. [pp. 28- 29] for examples). Deviation models must also be adjusted so that the deviations are weighted by their probability as done in the MOTAD version of the discrete stochastic model in section 14.23.

A third and again independent line of comment relates to the question "should uncertainty be modeled and if so, how?" Such a concern is paramount to this section. It is obvious from the above that in modeling uncertainty, data are needed describing the uncertainty, and that modeling uncertainty makes a model larger and more complex, and therefore harder to interpret, explain, and deal with. It is not the purpose of these comments to resolve this question, but rather to enter some considerations to the resolution of this question. First and fundamentally, if a model solution diverges from reality because the decision maker in reality has somehow considered risk, then it is important to consider risk. This leads to the subjective judgment on behalf of the modeling team as to whether risk makes a difference. Given that risk is felt to make a difference, then, how should risk be modeled? In the approaches above, the formulation model depends upon whether there is conditional decision making and on what is uncertain. These formulations are not mutually exclusive; rather, it may be desirable to use combinations of these formulations (see, for example, Wicks and Guise, Tice or Klemme).

Several uncertainty models have not been covered the above discussion. There are more advanced applications of chance constrained programming such as those found in the books by Sengupta; Vajda; and Kolbin. Another approach is called "Cautions Suboptimizing" by Day (1979). This approach bounds the adjustments in variables to a maximum amount in any one year. We also have not covered Monte Carlo programming as espoused by Anderson, et al., mainly because we do not feel it falls into the class of programming techniques but rather is a simulation technique.

Finally, it is relevant to discuss how risk should be modeled. There have been arguments presented in literature (e.g. see, for example, Baker and McCarl or Musser, et al.) that risk model solutions are biased if the model structure is not adequate before risk modeling is incorporated. Baker and McCarl argue that one should not include risk until the model structure is fully specified in terms of the needed constraints, the time disaggregation of constraints, and activities.

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Table 14.1. Data for E-V Example -- Returns by Stock and Event

	----Stock Returns by Stock and Event----			
	Stock1	Stock2	Stock3	Stock4
Event1	7	6	8	5
Event2	8	4	16	6
Event3	4	8	14	6
Event4	5	9	-2	7
Event5	6	7	13	6
Event6	3	10	11	5
Event7	2	12	-2	6
Event8	5	4	18	6
Event9	4	7	12	5
Event10	3	9	-5	6
	Stock1	Stock2	Stock3	Stock4
Price	22	30	28	26

Table 14.2. Mean Returns and Variance Parameters for Stock Example

	Stock1	Stock2	Stock3	Stock4
Mean Returns	4.70	7.60	8.30	5.80
Variance-Covariance Matrix				
	Stock1	Stock2	Stock3	Stock4
Stock1	3.21	-3.52	6.99	0.04
Stock2	-3.52	5.84	-13.68	0.12
Stock3	6.99	-13.68	61.81	-1.64
Stock4	0.04	0.12	-1.64	0.36

Table 14.3. GAMS Formulation of E-V Problem

```
SETS       STOCKS  POTENTIAL INVESTMENTS / BUYSTOCK1*BUYSTOCK4 /
           EVENTS  EQUALLY LIKELY RETURN STATES OF NATURE
                                         /EVENT1*EVENT10 / ;

ALIAS (STOCKS,STOCK);

PARAMETERS     PRICES(STOCKS) PURCHASE PRICES OF THE STOCKS
                             / BUYSTOCK1   22
                               BUYSTOCK2   30
                               BUYSTOCK3   28
                               BUYSTOCK4   26 / ;

SCALAR      FUNDS    TOTAL INVESTABLE FUNDS / 500 / ;

TABLE RETURNS(EVENTS,STOCKS) RETURNS BY STATE OF NATURE EVENT

          BUYSTOCK1  BUYSTOCK2   BUYSTOCK3   BUYSTOCK4
  EVENT1      7          6           8           5
  EVENT2      8          4          16           6
  EVENT3      4          8          14           6
  EVENT4      5          9          -2           7
  EVENT5      6          7          13           6
  EVENT6      3         10          11           5
  EVENT7      2         12          -2           6
  EVENT8      5          4          18           6
  EVENT9      4          7          12           5
  EVENT10     3          9          -5           6

PARAMETERS
   MEAN (STOCKS)       MEAN RETURNS TO X(STOCKS)
   COVAR(STOCK,STOCKS) VARIANCE COVARIANCE MATRIX;

MEAN(STOCKS) = SUM(EVENTS , RETURNS(EVENTS,STOCKS) / CARD(EVENTS) );
COVAR(STOCK,STOCKS)
     = SUM (EVENTS ,(RETURNS(EVENTS,STOCKS) - MEAN(STOCKS))
                   *(RETURNS(EVENTS,STOCK)- MEAN(STOCK)))/CARD(EVENTS);

DISPLAY MEAN , COVAR ;

SCALAR RAP   RISK AVERSION PARAMETER / 0.0 / ;

POSITIVE VARIABLES    INVEST(STOCKS)  MONEY INVESTED IN EACH STOCK

VARIABLE                      OBJ            NUMBER TO BE MAXIMIZED ;

EQUATIONS                     OBJJ           OBJECTIVE FUNCTION
                                  INVESTAV      INVESTMENT FUNDS AVAILABLE
           ;

OBJJ..
OBJ =E=   SUM(STOCKS, MEAN(STOCKS) * INVEST(STOCKS))
        - RAP*(SUM(STOCK, SUM(STOCKS,
           INVEST(STOCK)* COVAR(STOCK,STOCKS) * INVEST(STOCKS))));

INVESTAV..     SUM(STOCKS, PRICES(STOCKS) * INVEST(STOCKS)) =L= FUNDS ;

MODEL EVPORTFOL /ALL/ ;

SOLVE EVPORTFOL USING NLP MAXIMIZING OBJ ;

SCALER VAR  THE VARIANCE ;
       VAR = SUM(STOCK, SUM(STOCKS,
            INVEST.L(STOCK)* COVAR(STOCK,STOCKS) * INVEST.L(STOCKS))) ;
```



```
DISPLAY VAR ;
```

```
SET RAPS   RISK AVERSION PARAMETERS /R0*R25/

PARAMETER RISKAVER(RAPS) RISK AVERSION COEFICIENTS
          /R0   0.00000,  R1   0.00025,  R2   0.00050,  R3   0.00075,
           R4   0.00100,  R5   0.00150,  R6   0.00200,  R7   0.00300,
           R8   0.00500,  R9   0.01000,  R10  0.01100,  R11  0.01250,
           R12  0.01500,  R13  0.02500,  R14  0.05000,  R15  0.10000,
           R16  0.30000,  R17  0.50000,  R18  1.00000,  R19  2.50000,
           R20  5.00000,  R21  10.0000,  R22  15.    ,  R23  20.
           R24  40.    ,  R25  80./

PARAMETER OUTPUT(*,RAPS) RESULTS FROM MODEL RUNS WITH VARYING RAP

OPTION SOLPRINT = OFF;

LOOP (RAPS,RAP=RISKAVER(RAPS);
       SOLVE EVPORTFOL USING NLP MAXIMIZING OBJ ;
       VAR = SUM(STOCK, SUM(STOCKS,
           INVEST.L(STOCK)* COVAR(STOCK,STOCKS) * INVEST.L(STOCKS))) ;
       OUTPUT("OBJ",RAPS)=OBJ.L;
       OUTPUT("RAP",RAPS)=RAP;
       OUTPUT(STOCKS,RAPS)=INVEST.L(STOCKS);
       OUTPUT("MEAN",RAPS)=SUM(STOCKS, MEAN(STOCKS)*INVEST.L(STOCKS));
       OUTPUT("VAR",RAPS) = VAR;
       OUTPUT("STD",RAPS)=SQRT(VAR);
       OUTPUT("SHADPRICE",RAPS)=INVESTAV.M;
       OUTPUT("IDLE",RAPS)=FUNDS-INVESTAV.L
             );
DISPLAY OUTPUT;
```

Table 14.4. E-V Example Solutions for Alternative Risk Aversion Parameters

RAP	0	0.00025	0.0005	0.00075	0.001
BUYSTOCK2			1.263	5.324	7.355
BUYSTOCK3	17.857	17.857	16.504	12.152	9.977
OBJ	148.214	143.287	138.444	135.688	134.245
MEAN	148.214	148.214	146.581	141.331	138.705
VAR	19709.821	19709.821	16274.764	7523.441	4460.478
STD	140.392	140.392	127.573	86.738	66.787
SHADPRICE	0.296	0.277	0.261	0.260	0.260
RAP	0.0015	0.002	0.003	0.005	0.010
BUYSTOCK2	9.386	10.401	11.416	12.229	12.838
BUYSTOCK3	7.801	6.713	5.625	4.755	4.102
OBJ	132.671	131.753	130.575	129.005	125.999
MEAN	136.080	134.767	133.454	132.404	131.617
VAR	2272.647	1506.907	959.949	679.907	561.764
STD	47.672	38.819	30.983	26.075	23.702
SHADPRICE	0.259	0.257	0.255	0.251	0.241
RAP	0.011	0.012	0.015	0.025	0.050
BUYSTOCK1			1.273	4.372	4.405
BUYSTOCK2	12.893	12.960	12.420	11.070	8.188
BUYSTOCK3	4.043	3.972	3.550	2.561	1.753
BUYSTOCK4					4.168
OBJ	125.441	124.614	123.380	120.375	116.805
MEAN	131.545	131.459	129.839	125.939	121.656
VAR	554.929	547.587	430.560	222.576	97.026
STD	23.557	23.401	20.750	14.919	9.850
SHADPRICE	0.239	0.236	0.234	0.230	0.224
RAP	0.100	0.300	0.500	1.000	2.500
BUYSTOCK1	4.105	3.905	3.865	3.835	1.777
BUYSTOCK2	6.488	5.354	5.128	4.958	2.289
BUYSTOCK3	1.340	1.064	1.009	0.968	0.446
BUYSTOCK4	6.829	8.602	8.957	9.223	4.296
OBJ	113.118	102.254	92.010	66.674	27.185
MEAN	119.327	117.774	117.463	117.230	54.370
VAR	62.086	51.734	50.905	50.556	10.874
STD	7.879	7.193	7.135	7.110	3.298
SHADPRICE	0.214	0.173	0.133	0.032	0
IDLE FUNDS					268.044

Notes: RAP is the risk aversion parameter (Φ) value
Stocki gives the amount invested in stocki
Obj gives the objective function value
Mean gives expected income
Var gives the variance of income
STD gives the standard deviation of income
Shadprice gives the shadow price on the capital available constraint

Table 14.5. Deviations from the Mean for Portfolio Example

	Stock1	Stock2	Stock3	Stock4
Event1	2.3	-1.6	-0.3	-0.8
Event2	3.3	-3.6	7.7	0.2
Event3	-0.7	0.4	5.7	0.2
Event4	0.3	1.4	-10.3	1.2
Event5	1.3	-0.6	4.7	0.2
Event6	-1.7	2.4	2.7	-0.8
Event7	-2.7	4.4	-10.3	0.2
Event8	0.3	-3.6	9.7	0.2
Event9	-0.7	-0.6	3.7	-0.8
Event10	-1.7	1.4	-13.3	0.2

Table 14.6. Example MOTAD Model Formulation

$$
\begin{array}{lrrrrllcl}
\text{Max} & 4.70\ X_1 & +\ 7.60\ X_2 & +\ 8.30\ X_3 & +\ 5.80\ X_4 & & & -\ \gamma\ \sigma & \\
\text{s.t.} & 22\ X_1 & +\ 30\ X_2 & +\ 28\ X_3 & +\ 26\ X_4 & & & & \leq\ 500 \\
 & +2.300\ X_1 & -1.600\ X_2 & -0.300\ X_3 & -0.800\ X_4 & +d_1^- & & & \geq\ 0 \\
 & +3.300\ X_1 & -3.600\ X_2 & +7.700\ X_3 & +0.200\ X_4 & +d_2^- & & & \geq\ 0 \\
 & -0.700\ X_1 & +0.400\ X_2 & +5.700\ X_3 & +0.200\ X_4 & +d_3^- & & & \geq\ 0 \\
 & +0.300\ X_1 & +1.400\ X_2 & -10.300\ X_3 & +1.200\ X_4 & +d_4^- & & & \geq\ 0 \\
 & +1.300\ X_1 & -0.600\ X_2 & +4.700\ X_3 & +0.200\ X_4 & +d_5^- & & & \geq\ 0 \\
 & -1.700\ X_1 & +2.400\ X_2 & +2.700\ X_3 & -0.800\ X_4 & +d_6^- & & & \geq\ 0 \\
 & -2.700\ X_1 & +4.400\ X_2 & -10.300\ X_3 & +0.200\ X_4 & +d_7^- & & & \geq\ 0 \\
 & +0.300\ X_1 & -3.600\ X_2 & +9.700\ X_3 & +0.200\ X_4 & +d_8^- & & & \geq\ 0 \\
 & -0.700\ X_1 & -0.600\ X_2 & +3.700\ X_3 & -0.800\ X_4 & +d_9^- & & & \geq\ 0 \\
 & -1.700\ X_1 & +1.400\ X_2 & -13.300\ X_3 & +0.200\ X_4 & +d_{10}^- & & & \geq\ 0 \\
 & & & & & \sum_k d_k^- & -\ \text{TND} & & =\ 0 \\
 & & & & & & \Delta\ \text{TND} & -\ \sigma & =\ 0
\end{array}
$$

Table 14.7. MOTAD Example Solutions for Alternative Risk Aversion Parameters

RAP		0.050	0.100	0.110	0.120
BUYSTOCK2					11.603
BUYSTOCK3	17.857	17.857	17.857	17.857	5.425
OBJ	148.214	140.146	132.078	130.464	129.390
MEAN	148.214	148.214	148.214	148.214	133.213
MAD	122.143	122.143	122.143	122.143	24.111
STDAPPROX	161.367	161.367	161.367	161.367	31.854
VAR	19709.821	19709.821	19709.821	19709.821	883.113
STD	140.392	140.392	140.392	140.392	29.717
SHADPRICE	0.296	0.280	0.264	0.261	0.259

RAP	0.130	0.150	0.260	0.400	0.500
BUYSTOCK1					2.663
BUYSTOCK2	11.603	11.603	11.916	12.379	10.985
BUYSTOCK3	5.425	5.425	5.090	4.594	3.995
OBJ	129.072	128.435	125.179	121.204	118.606
MEAN	133.213	133.213	132.809	132.210	129.161
MAD	24.111	24.111	22.212	20.827	15.979
STDAPPROX	31.854	31.854	29.345	27.515	21.110
VAR	883.113	883.113	771.228	643.507	455.983
STD	29.717	29.717	27.771	25.367	21.354
SHADPRICE	0.258	0.257	0.250	0.242	0.237

RAP	0.750	1.000	1.250	1.500	1.750
BUYSTOCK1	5.145	7.119	2.817	2.817	2.817
BUYSTOCK2	10.409	9.879	5.617	5.617	5.617
BUYSTOCK3	2.661	1.564	1.824	1.824	1.824
BUYSTOCK4		0.123	8.402	8.402	8.402
OBJ	114.168	111.009	108.372	106.086	103.801
MEAN	125.384	122.240	119.799	119.799	119.799
MAD	11.320	8.501	6.920	6.920	6.920
STDAPPROX	14.955	11.231	9.142	9.142	9.142
VAR	211.996	121.386	83.886	83.886	83.886
STD	14.560	11.018	9.159	9.159	9.159
SHADPRICE	0.228	0.222	0.217	0.212	0.208

RAP	2.000	2.500	5.000	10.000	12.500
BUYSTOCK1	2.817	2.817	2.858	2.858	2.858
BUYSTOCK2	5.617	5.617	4.178	4.178	4.178
BUYSTOCK3	1.824	1.824	1.242	1.242	1.242
BUYSTOCK4	8.402	8.402	10.654	10.654	10.654
OBJ	101.515	96.944	76.540	35.790	15.415
MEAN	119.799	119.799	117.289	117.289	117.289
MAD	6.920	6.920	6.169	6.169	6.169
STDAPPROX	9.142	9.142	8.150	8.150	8.150
VAR	83.886	83.886	57.695	57.695	57.695
STD	9.159	9.159	7.596	7.596	7.596
SHADPRICE	0.203	0.194	0.153	0.072	0.031

Note: The abbreviations are the same as in Table 14.4 with the addition of MAD which gives the mean absolute deviation and STDAPPROX which gives the standard deviation approximation.

Table 14.8. Example Formulation of Safety First Problem

$$
\begin{array}{llllllllll}
\text{Max} & 4.70\ X_1 & + & 7.60\ X_2 & + & 8.30\ X_3 & + & 5.80\ X_4 & & \\
\text{s.t.} & 22\ X_1 & + & 30\ X_2 & + & 28\ X_3 & + & 26\ X_4 & \leq & 500 \\
 & 7\ X_1 & + & 6\ X_2 & + & 8\ X_3 & + & 5\ X_4 & \geq & S \\
 & 8\ X_1 & + & 4\ X_2 & + & 16\ X_3 & + & 6\ X_4 & \geq & S \\
 & 4\ X_1 & + & 8\ X_2 & + & 14\ X_3 & + & 6\ X_4 & \geq & S \\
 & 5\ X_1 & + & 9\ X_2 & - & 2\ X_3 & + & 7\ X_4 & \geq & S \\
 & 6\ X_1 & + & 7\ X_2 & + & 13\ X_3 & + & 6\ X_4 & \geq & S \\
 & 3\ X_1 & + & 10\ X_2 & + & 11\ X_3 & + & 5\ X_4 & \geq & S \\
 & 2\ X_1 & + & 12\ X_2 & - & 2\ X_3 & + & 6\ X_4 & \geq & S \\
 & 5\ X_1 & + & 4\ X_2 & + & 18\ X_3 & + & 6\ X_4 & \geq & S \\
 & 4\ X_1 & + & 7\ X_2 & + & 12\ X_3 & + & 5\ X_4 & \geq & S \\
 & 3\ X_1 & + & 9\ X_2 & - & 5\ X_3 & + & 6\ X_4 & \geq & S
\end{array}
$$

Table 14.9. Safety First Example Solutions for Alternative Safety Levels

RUIN	-100.000	-50.000	0.0	25.000	50.000
BUYSTOCK2	0.0	2.736	6.219	7.960	9.701
BUYSTOCK3	17.857	14.925	11.194	9.328	7.463
OBJ	148.214	144.677	140.174	137.923	135.672
MEAN	148.214	144.677	140.174	137.923	135.672
VAR	19709.821	12695.542	6066.388	3717.016	2011.116
STD	140.392	112.674	77.887	60.967	44.845
SHADPRICE	0.296	0.280	0.280	0.280	0.280

Note: The abbreviations are the same as in the previous example solutions with RUIN giving the safety level.

Table 14.10. Example Formulation of Target MOTAD Problem

$$
\begin{array}{llllllllllllll}
\text{Max} & 4.70\ X_1 & + & 7.60\ X_2 & + & 8.30\ X_3 & + & 5.80\ X_4 & & & & \\
\text{s.t.} & 22\ X_1 & + & 30\ X_2 & + & 28\ X_3 & + & 26\ X_4 & & & \leq & 500 \\
 & 7\ X_1 & + & 6\ X_2 & + & 8\ X_3 & + & 5\ X_4 & + & Dev_1 & \geq & T \\
 & 8\ X_1 & + & 4\ X_2 & + & 16\ X_3 & + & 6\ X_4 & + & Dev_2 & \geq & T \\
 & 4\ X_1 & + & 8\ X_2 & + & 14\ X_3 & + & 6\ X_4 & + & Dev_3 & \geq & T \\
 & 5\ X_1 & + & 9\ X_2 & - & 2\ X_3 & + & 7\ X_4 & + & Dev_4 & \geq & T \\
 & 6\ X_1 & + & 7\ X_2 & + & 13\ X_3 & + & 6\ X_4 & + & Dev_5 & \geq & T \\
 & 3\ X_1 & + & 10\ X_2 & + & 11\ X_3 & + & 5\ X_4 & + & Dev_6 & \geq & T \\
 & 2\ X_1 & + & 12\ X_2 & - & 2\ X_3 & + & 6\ X_4 & + & Dev_7 & \geq & T \\
 & 5\ X_1 & + & 4\ X_2 & + & 18\ X_3 & + & 6\ X_4 & + & Dev_8 & \geq & T \\
 & 4\ X_1 & + & 7\ X_2 & + & 12\ X_3 & + & 5\ X_4 & + & Dev_9 & \geq & T \\
 & 3\ X_1 & + & 9\ X_2 & - & 5\ X_3 & + & 6\ X_4 & + & Dev_{10} & \geq & T \\
 & & & & & & & & & \sum_k Dev_k & \leq & \lambda
\end{array}
$$

Table 14.11. Target MOTAD Example Solutions for Alternative Deviation Limits

TARGETDEV	120.000	60.000	24.000	12.000	10.800
BUYSTOCK2	0.0	0.0	7.081	10.193	10.516
BUYSTOCK3	17.857	17.857	10.270	6.936	6.590
OBJ	148.214	148.214	139.059	135.037	134.618
MEAN	148.214	148.214	139.059	135.037	134.618
VAR	19709.821	19709.821	4822.705	1646.270	1433.820
STD	140.392	140.392	69.446	40.574	37.866
SHADPRICE	0.296	0.296	0.286	0.295	0.295

TARGETDEV	8.400	7.200	3.600
BUYSTOCK1	0.0	0.0	3.459
BUYSTOCK2	11.259	11.782	11.405
BUYSTOCK3	5.794	5.234	2.919
OBJ	133.659	132.982	127.168
MEAN	133.659	132.982	127.168
VAR	1030.649	816.629	277.270
STD	32.104	28.577	16.651
SHADPRICE	0.298	0.298	0.815

Note: The abbreviations are again the same with TARGETDEV giving the λ value.

Table 14.12. Example Formulation of DEMP Problem

$$\text{Max} \sum_k (W_k)^{power}$$

$$
\begin{array}{lrrrrrrrrr}
\text{s.t.} & & 22\,X_1 & + & 30\,X_2 & + & 28\,X_3 & + & 26\,X_4 & \leq 500 \\
W_1 & - & 7\,X_1 & - & 6\,X_2 & - & 8\,X_3 & - & 5\,X_4 & = 100 \\
W_2 & - & 8\,X_1 & - & 4\,X_2 & - & 16\,X_3 & - & 6\,X_4 & = 100 \\
W_3 & - & 4\,X_1 & - & 8\,X_2 & - & 14\,X_3 & - & 6\,X_4 & = 100 \\
W_4 & - & 5\,X_1 & - & 9\,X_2 & + & 2\,X_3 & - & 7\,X_4 & = 100 \\
W_5 & - & 6\,X_1 & - & 7\,X_2 & - & 13\,X_3 & - & 6\,X_4 & = 100 \\
W_6 & - & 3\,X_1 & - & 10\,X_2 & - & 11\,X_3 & - & 5\,X_4 & = 100 \\
W_7 & - & 2\,X_1 & - & 12\,X_2 & + & 2\,X_3 & - & 6\,X_4 & = 100 \\
W_8 & - & 5\,X_1 & - & 4\,X_2 & - & 18\,X_3 & - & 6\,X_4 & = 100 \\
W_9 & - & 4\,X_1 & - & 7\,X_2 & - & 12\,X_3 & - & 5\,X_4 & = 100 \\
W_{10} & - & 3\,X_1 & - & 9\,X_2 & + & 5\,X_3 & - & 6\,X_4 & = 100
\end{array}
$$

Table 14.13. DEMP Example Solutions for Alternative Utility Function Exponents

POWER	0.950	0.900	0.750	0.500	0.400
BUYSTOCK2			4.560	8.563	9.344
BUYSTOCK3	17.857	17.857	12.972	8.683	7.846
OBJ	186.473	140.169	60.363	15.282	8.848
MEAN	248.214	248.214	242.319	237.144	236.134
VAR	19709.821	19709.821	8903.295	3054.034	2309.233
STD	140.392	140.392	94.357	55.263	48.054
SHADPRICE	0.287	0.277	0.269	0.266	0.265
POWER	0.300	0.200	0.100	0.050	0.030
BUYSTOCK2	9.919	10.358	10.705	10.852	10.907
BUYSTOCK3	7.230	6.759	6.388	6.230	6.171
OBJ	5.127	2.972	1.724	1.313	1.177
MEAN	235.390	234.822	234.374	234.184	234.113
VAR	1843.171	1534.736	1320.345	1236.951	1207.076
STD	42.932	39.176	36.337	35.170	34.743
SHADPRICE	0.264	0.264	0.263	0.263	0.263
POWER	0.020	0.010	0.001	0.0001	
BUYSTOCK2	10.934	10.960	10.960	10.960	
BUYSTOCK3	6.143	6.115	6.115	6.115	
OBJ	1.115	1.056	1.005	1.001	
MEAN	234.079	234.045	234.045	234.045	
VAR	1192.805	1178.961	1178.961	1178.961	
STD	34.537	34.336	34.336	34.336	
SHADPRICE	0.263	0.263	0.263	0	

Note: The abbreviations are again the same with POWER giving the exponent used.

Table 14.14. Chance Constrained Example Data

Event	Small Lathe	Large Lathe	Carver
1	140	90	120
2	120	94	132
3	133	88	110
4	154	97	118
5	133	87	133
6	142	86	107
7	155	90	120
8	140	94	114
9	142	89	123
10	141	85	123
Mean	140	90	120
Standard Error	9.63	3.69	8.00

Table 14.15. Chance Constrained Example Solutions for Alternative Alpha Levels

Z_α	0.00	1.280	1.654	2.330
PROFIT	10417.291	9884.611	9728.969	9447.647
SMLLATHE	140.000	127.669	124.067	117.554
LRGLATHE	90.000	85.280	83.900	81.407
CARVER	120.000	109.760	106.768	101.360
LABOR	125.000	125.000	125.000	125.000
FUNCTNORM	62.233	78.102	82.739	91.120
FANCYNORM	73.020	51.495	45.205	33.837
FANCYMXLRG	5.180	6.788	7.258	8.108

Note: Z_α is the risk aversion parameter.

Table 14.16. Feed Nutrients by State of Nature for Wicks Guise Example

Nutrient	State	CORN	SOYBEANS	WHEAT
ENERGY	S1	1.15	0.26	1.05
ENERGY	S2	1.10	0.31	0.95
ENERGY	S3	1.25	0.23	1.08
ENERGY	S4	1.18	0.28	1.12
PROTEIN	S1	0.23	1.12	0.51
PROTEIN	S2	0.17	1.08	0.59
PROTEIN	S3	0.25	1.01	0.46
PROTEIN	S4	0.15	0.99	0.56

Table 14.17. Wicks Guise Example

	Corn	Soybeans	Wheat	EnDev	EnMAD	Enσ	PrDev	PrMAD	Prσ		
Objective	0.03	0.06	0.04								
Volume	1	1	1							=	1
Energy	1.17	0.27	1.05			$-\phi$				≥	0.8
Protein	0.20	1.05	0.53						$-\phi$	≥	0.5
Energys1	−0.02	−0.01	+0.00	$-d_{e1}^{+}+d_{e1}^{-}$						=	0
Energys2	−0.07	+0.04	−0.10	$-d_{e2}^{+}+d_{e2}^{-}$						=	0
Energys3	+0.08	−0.04	+0.03	$-d_{e3}^{+}+d_{e3}^{-}$						=	0
Energys4	+0.01	+0.01	+0.07	$-d_{e4}^{+}+d_{e4}^{-}$						=	0
EnergyMAD				$\sum_k (d_{ek}^{+}+d_{ek}^{-})/4$	-1					=	0
Energyσ					$-\Delta$	$+1$				=	0
Proteins1	−0.02	−0.01	+0.00				$-d_{p1}^{+}+d_{p1}^{-}$			=	0
Proteins2	−0.07	+0.04	−0.10				$-d_{p2}^{+}+d_{p2}^{-}$			=	0
Proteins3	+0.08	−0.04	+0.03				$-d_{p3}^{+}+d_{p3}^{-}$			=	0
Proteins4	+0.01	+0.01	+0.07				$-d_{p4}^{+}+d_{p4}^{-}$			=	0
ProteinMAD							$\sum_k (d_{pk}^{+}+d_{pk}^{-})/4$	-1		=	0
Proteinσ								$-\Delta$	$+1$	=	0

Note: EnDev is the energy deviation
EnMAD is the energy mean absolute deviation
Enσ is the energy standard deviation approximations
PrDev is the protein deviation
PrMAD is the protein mean absolute deviation
Prσ is the protein standard deviation approximation

Table 14.18. Results From Example Wicks Guise Model Runs With Varying RAP

RAP		0.250	0.500	0.750	1.000
CORN	0.091	0.046	0.211	0.230	0.221
SOYBEANS			0.105	0.129	0.137
WHEAT	0.909	0.954	0.684	0.641	0.642
OBJ	0.039	0.040	0.040	0.040	0.041
AVGPROTEIN	0.500	0.515	0.515	0.521	0.529
STDPROTEIN	0.054	0.059	0.030	0.028	0.029
AVGENERGY	1.061	1.056	0.993	0.977	0.969
STDENERGY	0.072	0.072	0.061	0.059	0.058
SHADPROT	0.030	0.033	0.036	0.037	0.038
RAP	1.250	1.500	2.000		
CORN	0.211	0.200	0.177		
SOYBEANS	0.146	0.156	0.176		
WHEAT	0.643	0.644	0.647		
OBJ	0.041	0.041	0.042		
AVGPROTEIN	0.536	0.545	0.563		
STDPROTEIN	0.029	0.030	0.031		
AVGENERGY	0.961	0.953	0.934		
STDENERGY	0.057	0.056	0.055		
SHADPROT	0.039	0.040	0.042		

Note: RAP gives the risk aversion parameter used
CORN gives the amount of corn used in the solution
SOYBEANS gives the amount of soybeans used in the solution
WHEAT gives the amount of wheat used in the solution
OBJ gives the objective function value
AVGPROTEIN gives the average amount of protein in the diet
STDPROTEIN gives the standard error of protein in the diet
AVGENERGY gives the average amount of energy in the diet
STDENERGY gives the standard error of energy in the diet
SHADPROT gives the shadow price on the protein requirement constraint

Table 14.19. Data on Uncertain Parameters in First SPR Example

	Value Under	
Parameter	State of Nature 1	State of Nature 2
Probability	.6	.4
Corn Yield in bu	100	105
Wheat Yield in bu	40	38
Corn Harvest Rate hours per bu	.010	.015
Wheat Harvest Rate hours per bu	.030	.034
Corn Price per bu	2.25	2.00
Wheat Price per bu	5.00	6.00
Harvest Time hours	122	143

Table 14.20. Risk Free Formulation of First SPR Example

	Grow Corn	Grow Wheat	Income	Harvest Corn	Harvest Wheat		
Objective			1				
Land	1	1				≤	100
Corn Yield Balance	$-yield_c$			1		≤	0
Wheat Yield Balance		$-yield_w$			1	≤	0
Harvest Hours				$+harvtime_c$	$+harvtime_w$	≤	harvavail
Income	-100	-60	-1	$+price_c$	$+price_w$	=	0

Table 14.21. Formulation of First SPR Example

		Grow Corn	Grow Wht.	Inc. s1	Harv Corn s1	Harv Wht s1	Inc. s2	Harv Corn s2	Harv Wht s2		RHS
				State 1			State 2				
	Objective			.6			.4				max
	Land	1	1							≤	100
State 1	Corn s1	-100			1					≤	0
	Wheat s1		-40			1				≤	0
	Harvest Hours s1				.010	.030				≤	122
	Income s1	-100	-60	-1	2.25	5.00				=	0
State 2	Corn s2	-105						1		≤	0
	Wheat s2		-38						1	≤	0
	Harvest Hours s2							.015	.034	≤	143
	Income s2	-100	-60				-1	2.00	6.00	=	0

Table 14.22. Solution of First SPR Example

Equation	Slack	Shadow Price
Objective	21340	
Land	0	175.59
Corn s1	0	-1.35
Wheat s1	0	-3.00
Harvest Hours s1	11.75	0
Income s1	0	-0.6
Corn s2	0	-0.387
Wheat s2	0	-1.463
Harvest Hours s2	0	27.56
Income s2	0	-0.4

Variable	Solution Value	Marginal Cost
Grow Corn	48.8	0
Grow Wheat	51.2	0
Income S1	21059	0
Harvest Corn s1	4876	0
Harvest Wheat s1	2049	0
Income S2	21762	0
Harvest Corn s2	5120	0
Harvest Wheat s2	1947	0

Table 14.23. Second SPR Example Formulation (Partial Tableau)

	Corn	Soy	Wht	Avg Cost	Pos Prot Dev s1	Neg Prot Dev s1	Pos Eng Dev s1	Neg Eng Dev s1	Cost s1	Pos Cost Dev s1	Negs Cost Dev s1	Pos Prot Dev s2	Neg Prot Dev s2	Pos Eng Dev s2	Neg Eng Dev s2	Cost s2	Pos Cost Dev s2	Neg Cost Dev s2		
Objective				1						+	+						+	+		
Total Feed	1	1	1																=	1
Average Cost				1					-.25							-.25			=	0
Protein-s1	0.23	1.12	0.51		-1	1													=	0.6
Energy -s1	1.15	0.26	1.05				-1	1											=	0.9
Cost-s1	0.03	0.06	0.04		0.50	1.50	1.00	0.10	-1										=	0
Cost dev s1				-1					1	-1	1								=	0
Protein-s2	0.17	1.08	0.59									-1	1						=	0.6
Energy -s2	1.10	0.31	0.95											-1	1				=	0.9
Cost-s2	.03	.06	.04									0.50	1.50	1.00	0.10	-1			=	0
Cost dev s2				-1												1	-1	1	=	0

Table 14.24. Second SPR Example Risk Neutral Solution

	Slack	Shadow Price		Slack	Shadow Price
Objective	0.067		Corn Purchase	0.283	0
Total Feed	0	-0.14	Soybean Purchase	0.362	0
Average Cost	0.00	1.	Wheat Purchase	0.355	0
Protein-s1	0	0.125	Average Cost	0.067	0
Energy -s1	0	0.025	Pos Protein Dev s1	0.052	0
Cost-s1	0	252.66	Neg Protein Dev s1	0.	0.50
Cost dev s1	0	0.00	Pos Energyn Dev s1	0.00	0
Protein-s2	0	0.125	Neg Energy Dev s1	0.108	0
Energy -s2	0	0.025	Cost - s1	0.081	0
Cost-s2	0	0.25	Pos Cost Dev - s1	0.014	0
Cost dev s2	0	0	Neg Cost Dev - s1	0.00	0
Protein-s3	0	-.366	Pos Protein Dev s2	0.049	0
Energy -s3	0	0.025	Neg Protein Dev s2	0.000	0.50
Cost-s3	0	0.25	Pos Energyn Dev s2	0.	0.275
Cost dev s3	0	0	Neg Energyn Dev s2	0.140	0
Protein-s4	0	.08	Cost - s2	0.083	0
Energy -s4	0	.025	Pos Cost Dev - s2	.016	0
Cost-s4	0	0.25	Neg Cost Dev - s2	0.00	0
Cost dev s4	0	0.00	Pos Protein Dev s3	0.	0.491
			Neg Protein Dev s3	0.	0.009
			Pos Energy Dev s3		0.275
			Neg Energy Dev s3	0.080	0
			Cost - s3	0.052	0
			Pos Cost Dev - s3	0.00	0
			Neg Cost Dev - s3	0.014	0
			Pos Protein Dev s4	0.	0.205
			Neg Protein Dev s4	0.	0.295
			Pos Energyn Dev s4	0.	0.275
			Neg Energy Dev s4	0.067	0
			Cost - s4	0.051	0
			Pos Cost Dev - s4	0.	0
			Neg Cost Dev - s4	0.016	0

Table 14.25. SPR Second Example Problem Soution Under Varying Risk Aversion

RAP	0	0.1	0.2	0.3	0.4	0.500	0.600
Corn	0.283	0.249	0.245	0.244	0.288	0.296	0.297
Soybeans	0.362	0.330	0.327	0.326	0.340	0.342	0.342
Wheat	0.355	0.422	0.428	0.430	0.372	0.363	0.361
Avgcost	0.067	0.067	0.067	0.067	0.071	0.071	0.071
Cost s1	0.081	0.074	0.073	0.073	0.071	0.071	0.071
Cost s2	0.083	0.080	0.080	0.080	0.074	0.073	0.073
Cost s3	0.052	0.066	0.067	0.068	0.071	0.071	0.071
Cost s4	0.051	0.048	0.048	0.048	0.067	0.070	0.071
Std Error	0.015	0.012	0.012	0.012	0.002	0.001	0.001

RAP is the risk aversion parameter.

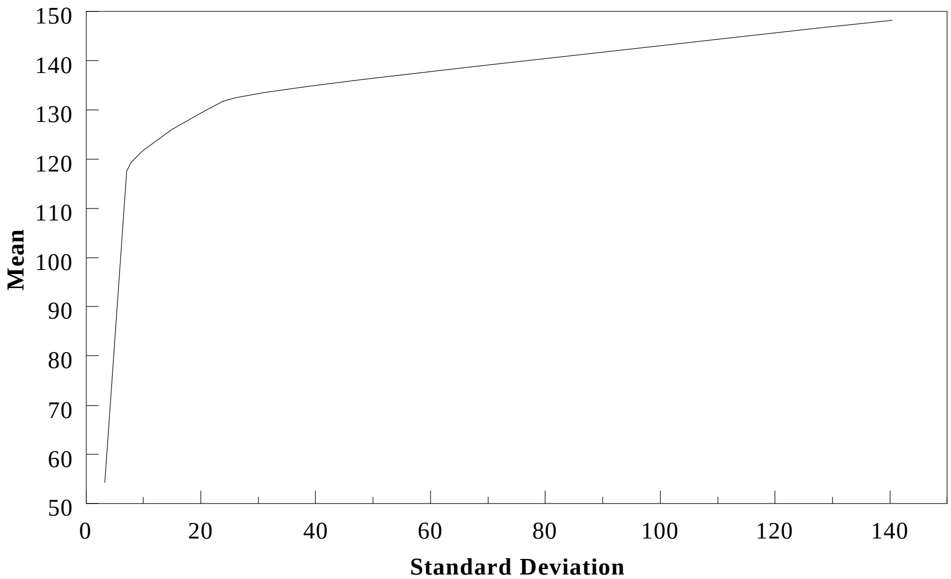
Table 14.26. Example Tableau for Third SPR Problem

	Average Ending Net Worth	Period 1		Period 2				Period 3									
								Period 2		Period 2		Period 2		Period 2			
				State 1		State 2		State 1		State 1		State 2		State 2			
								State A		State B		State A		State B			
		Sell	Keep	Sell	Keep	Sell	Keep	Sell	End Worth	Sell	End Worth	Sell	End Worth	Sell	End Worth		
Objective	1															max	
Starting Stock		1	1													≤	100
Avg End Worth	1								-0.42		-0.28		-0.21		-0.09	=	0
Stock Kept pd 1 to 2 s1			-1	1	1											≤	0
Stock Kept pd 1 to 2 s2			-1			1	1									≤	0
Stock Kept pd 2 to 3 s1-sA					-1			1								≤	0
Ending Worth s1-sA		2.1412		2.332				2.18	-1							=	0
Stock Kept pd 2 to 3 s1-sB					-1					1						≤	0
Ending Worth s1-sB		2.1008		2.288						2.44	-1					=	0
Stock Kept pd 2 to 3 s2-sA							-1					1				≤	0
Ending Worth s2-sA		2.1828				2.193						2.18	-1			=	0
Stock Kept pd 2 to 3 s2-sB							-1							1		≤	0
Ending Worth s2-sB		2.1012				2.111								2.44	-1	=	0

Table 14.27. Solution for Third SPR Example

Variable	Value	Reduced Cost	Variable	Slack	Shadow Price
Average Ending Net Worth	229.748	0	Objective	229.748	
Sell In Period 1	0	-0.162	Starting Stock	0	2.297
Keep From Period 1 to 2	100	0	Avg End Worth	0	1
Sell In Period 2 Under State 1	100	0	Stock Kept pd 1 to 2 s1	0	1.62
Keep From Period 2 to 3 Under State 1	0	-0.021	Stock Kept pd 1 to 2 s1	0	0.677
Sell In Period 2 Under State 2	0	-0.027	Stock Kept pd 2 to 3 s1-s1	0	0.916
Keep From Period 2 to 3 Under State 2	100	0	Ending Worth s1-s1	0	-0.42
Sell in Period 3 Under State 1 -- State A	0	0	Stock Kept pd 2 to 3 s1-s2	0	0.683
Ending Worth Under State 1 -- State A	233.2	0	Ending Worth s1-s2	0	-0.28
Sell In Period 3 Under State 1 -- State B	0	0	Stock Kept pd 2 to 3 s2-s1	0	0.458
Ending Worth Under State 1 -- State B	228.8	0	Ending Worth s2-s1	0	-0.21
Sell In Period 3 Under State 2 -- State A	100	0	Stock Kept pd 2 to 3 s2-s2	0	0.22
Ending Worth Under State 2 -- State A	218	0	Ending Worth s2-s2	0	-0.09
Sell In Period 3 Under State 2 -- State B	100	0			
Ending Worth Under State 2 -- State B	244	0			

14.1. E-V Model Efficient Frontier



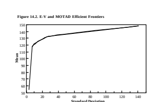


Figure 14.3.Decision Tree for Sequential Programming Example